Democracy Undone.

Systematic Minority Advantage in Competitive Vote Markets*

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Abstract

We study the competitive equilibrium of a market for votes where the choice is binary and it is known that a majority of the voters supports one of the two alternatives. Voters can trade votes for a numeraire before making a decision via majority rule. We identify a sufficient condition guaranteeing the existence of an ex ante equilibrium. In equilibrium, only the most intense voter on each side demands votes, and each demands enough votes to alone control a majority. The equilibrium strongly resembles an all-pay auction for decision power: it makes clear that votes are only a medium for the allocation of power. The probability of a minority victory is always higher than efficient and converges rapidly to one-half as the electorate increases, for any minority size. The numerical advantage of the majority becomes irrelevant: democracy is undone by the market.

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1 Introduction

In a broad sense, markets function well in allocating goods. Could they function well in allocating votes? Consider a group taking a single binary decision via majority voting. Majority voting ignores the intensity of voters' preferences, allowing an intense minority to lose to a tepid majority. In markets for goods, prices induce individuals to act according to the relative strength of their preferences. If majority voting were preceded by a market for votes, could intensity of preferences be expressed appropriately?

Markets for votes have long captured the imagination of philosophers, political scientists, and economists.[1] However, even ignoring ethical objections, finding a convincing answer to the question just raised has proved difficult. The problem is a fundamental non-convexity associated with vote trading: votes are intrinsically worthless, and their value depends on the holdings of votes by all other individuals. Thus, demands are interdependent, and payoffs discontinuous at the point at which a voter becomes pivotal. Both in a market for votes and in log-rolling games, traditional equilibrium concepts such as competitive equilibrium or the core typically fail to exist.[2]

Recently, a possible solution to the failure of equilibrium existence has been suggested. Focusing on a competitive market where voters can trade votes for a numeraire, Casella, Llorente-Saguer and Palfrey (2012) (CLP from now onward) propose the concept of *ex ante competitive equilibrium*: traders are allowed to express probabilistic demands and the market clears in expectation. At the equilibrium price, deviations from market clearing can occur, but they must be unsystematic and unexpected. Ex post, the imbalance between demand and supply is resolved by a rationing rule. CLP show that such an equilibrium exists in a symmetric model where each voter has equal probability of favoring either alternative, and where without vote trading the expected outcome of the vote is a tie.[3]

[1]Among economists and political scientists, the 1960's and 1970's in particular saw a large literature on the topic, whether studying trades of votes for votes or buying and selling of votes on a market in exchange for a numeraire. See: Buchanan and Tullock (1962), Coleman (1966, 1967), Park (1967), Wilson (1969), Tullock (1970), Haefele (1971), Kadane (1972), Riker and Brams (1973), Mueller (1967, 1973), Bernholtz (1973, 1974), Ferejohn (1974), Koehler (1975), Schwartz (1977). Among later contributions, see: Piketty (1994), Philipson and Snyder (1996), Kultti and Salonen (2005). For ethical and philosophical discussions of markets for votes, see, for example, Tobin (1970), Marshall (1977), Walzer (1983), Anderson (1993), Sandel (2012), and Satz (2012).

[2]Ferejohn (1974), (Schwartz (1977, 1981), Shubik and van der Heyden (1978), Weiss (1988), Philipson and Snyder (1996).

[3]Kultti and Salonen (2005) also propose a Walrasian approach to vote markets based on probabilistic demands, but do not impose any market clearing condition. Recently Iaryczower and Oliveros (2013) have

The result addresses the existence problem that has hampered the study of vote markets, and the concept of ex ante equilibrium is found to have good predictive power in a laboratory experiment. The symmetry assumption, however, is restrictive and not ideally suited to the question that motivated the research. What we want to know is whether minority voters can buy enough votes from the majority to overcome their numerical inferiority, *when both groups are aware of their minority and majority status.* The interest is not only theoretical. In most applications, the two opposing groups are well informed about which side is holding the majority: sides are not equal-sized and are well-established by party labels, cultural and geopolitical characteristics, or historical voting patterns. This is the environment we study in this paper.

The difficulty is that the more precise information exacerbates the non-convexity problem associated with votes. The literature has conjectured, plausibly, that any equilibrium in a market for votes requires uncertainty about the alternative preferred by a majority of the voters.[4] This paper studies and eventually contradicts this conjecture. In so doing, it establishes two general points. First, the obstacles to equilibrium existence in a competitive market for votes are logically unrelated to uncertainty about the direction of preferences. Indeed, our results hold identically under different informational assumptions, and under both complete and incomplete information, as long as voters know their own majority or minority status.[5] Second, the concept of ex ante competitive equilibrium generalizes to an asymmetric setting: the contribution in CLP is not limited to a knife-edge case. We construct an ex ante equilibrium that extends in intuitive fashion the equilibrium characterized by CLP.

We study a group of voters who take a single binary decision by majority voting. Before voting, individuals can buy and sell votes among themselves in a competitive market, in exchange for a numeraire. No individual is liquidity constrained. We obtain two main results. First, we identify a sufficient condition guaranteeing that an ex ante equilibrium with vote trading exists for arbitrary electorate size and majority/minority partition. The condition rules out the possibility that multiple members of one group all have preferences

proposed modeling vote-trading through decentralized bargaining.

[4]See for example Piketty (1994).

[5]Our results hold under complete information, when each voter's direction and intensity of preferences are publicly known. But they also hold if intensities of preferences are private information, and in this case they hold under different scenarios: when each voter's individual membership in the majority or minority is publicly known; when the sizes of the two groups are known, but other voters' individual membership is not, and they hold when voters know their own minority or majority status, but cannot estimate precisely the size of the two groups.

that are much more intense (in a precise sense) than any member of the opposite group. At small electorate sizes, we find that the equilibrium exists with high probability for standard intensity distributions–for example, if the minority is a third of the electorate and the distribution of intensities is uniform, the equilibrium exists with probability larger than 98 percent with nine voters, and larger than 99.9 percent with 21 voters. In large electorates, an ex ante equilibrium with trade exists with probability arbitrarily close to 1, for any intensity distribution and any minority share.

Second, the equilibrium we characterize has strong properties that translate into a systematic bias in favor of the minority, relative to the efficient outcome: for any electorate size, any majority/minority partition, and any distribution of intensities, the minority wins more frequently than efficiency dictates.

In equilibrium, only the highest intensity member of each group demands votes with positive probability; all other individuals offer their vote for sale. Of the two voters who are potential buyers, the voter belonging to the minority is weakly more aggressive: he may demand to buy with higher probability than the majority voter even when his intensity is lower. Together, these properties imply that the market works not only to weaken but to erase the advantage enjoyed by the majority.[6] Because all other voters offer to sell their votes, the two highest-intensity individuals must each demand enough votes to single-handedly control a majority. Their distinct status as minority or majority members becomes irrelevant. Again, this is particularly clear in large electorates. In such settings, the minority is always expected to win half of the time, for any distribution of intensities and *regardless of its share of the electorate*. As we summarize in the title of this paper: democracy–the power of majority rule–is undone by the market: the numerical superiority of the majority loses all its significance.

The market for votes always falls short of the first best. How it compares to majority voting with no trade depends on the shape of the distribution of intensities. In a small electorate, minority victories must reduce welfare if the wedge between the highest expected intensity and the average intensity is not too large. The result is intuitive: even when the highest value belongs to a minority voter, there is a trade-off between such a value and the values forgone by majority members who are, by definition, more numerous. In a large

[6] The result echoes the conclusion of costly voting models, where the "underdog effect" predicts higher turn-out rates by the minority (Simon (1954), Gartner (1976), Levine and Palfrey (2007)). Especially in large groups, the finding that participation can be small when it is costly is common to different models, for example Osborne et al.(2000).

electorate, the bias in favor of the minority is strong enough that ex ante welfare is always lower than in the absence of trade, for any distribution of intensities. Because the minority always wins with probability one half, the welfare loss is larger the smaller the minority size: the expected loss can be quantified precisely and is inversely related to the minority size.

The equilibrium we construct echoes the equilibrium in CLP: a vote market leads individuals to either demand a majority of votes or sell. The robustness of this finding to the existence of asymmetric, known groups with opposite preferences suggests that, by reestablishing existence, the concept of ex ante equilibrium allows us to tap into a deeper vein of economic intuition. Votes per se are worthless, and the market comes to resemble an auction for decision power. As in an all-pay auction, the competition is between the two individuals who have most to gain from controlling it, while the others refrain from participating (Hillman and Riley, 1989; Baye et al. 1996, Siegel, 2009). The aggregate values of the two opposing groups are not internalized and the final outcome is inefficient, but the market functions as we should have expected.

In addition to supporting this interpretation of a market for votes, the asymmetric model studied here delivers a number of novel predictions. First, because in both groups most individuals are offering their vote for sale, demand for additional votes is just as likely to arise from the majority as from the minority. Second, in equilibrium, intra-group trade and super-majorities always arise with high probability, even though votes command a positive price, the majority size is known, and all demands for votes are expressed simultaneously. The intuition is clear: high intensity individuals need to preempt sales to the opposite group by their own weak allies. We believe that the predictions are empirically very plausible, but absent from the models of intra-group vote-buying we are familiar with.[7]

Beyond its strict tie to the existing studies of vote markets, this paper is related to two other strands of literature. First, there is the important but different literature where candidates or lobbies buy voters' or legislators' votes: for example, Myerson (1993), Groseclose and Snyder (1996), Dal Bò (2007), Dekel, Jackson and Wolinsky (2008) and (2009). These papers differ from the problem we study because in our case vote trading happens *within* the committee (or the electorate). The individuals buying votes are members of the group themselves, and each individual is potentially both a buyer and a seller. This matters be-

[7]Groseclose and Snyder's (1996) conclusion that vote-buying leads to supermajorities has the same flavor but a different origin. Their paper studies vote-buying in a legislature by two competing outside buyers, as opposed to vote trading among voters, and their result is due to the buyers taking turns in proposing a deal to the legislators, as opposed to the one-shot market studied here.

cause it adds a public good aspect to vote trades: purchases of votes help all members of one's group and hurt all members of the opposite group.

Second, vote markets are not the only remedy advocated for majority rule's failure to recognize intensity of preferences in binary decisions. The mechanism design literature has proposed mechanisms with side payments, building on Groves-Clarke taxes (e.g., d'Apremont and Gerard-Varet 1979). However, these mechanisms have problems with bankruptcy, budget balance, and collusion (Green and Laffont 1979, Mailath and Postlewaite 1990). A recent literature suggests combining insights from mechanism design into the design of voting rules. Goeree and Zhang (2012) and Weyl (2012) propose allowing voters to purchase votes from a central agency at a price equal to the square of the number of votes purchased, a scheme with strongly desirable asymptotic properties. Casella (2005, 2012), Jackson and Sonnenschein (2007), and Hortala-Vallve (2012) propose mechanisms without transfer that allow agents to express the relative intensity of their preferences by linking decisions across issues. Casella, Gelman and Palfrey (2006), Casella, Palfrey and Riezman (2008), Engelmann and Grimm (2012), and Hortala-Vallve and Llorente-Saguer (2010) test the performance of these mechanisms experimentally and find that efficiency levels are very close to theoretical equilibrium predictions, even in the presence of some deviations from theoretical equilibrium strategies.

The rest of the paper is organized as follows. Section 2 presents the model; section 3 characterizes the ex ante equilibrium whose properties we discuss in the rest of the paper; section 4 studies the expected frequency of minority victories and expected welfare, and compares these measures to the equivalent measures in the absence of a vote market and in the utilitarian first best. Section 5 discusses the robustness of the results to alternative assumptions about information, the rationing rule and the stochastic process generating intensities. Section 6 concludes. The Appendix collects the proofs.

2 The Model

A committee of size n (odd) must decide between two alternatives, A and B. The committee is divided into two groups with opposite preferences: M individuals prefer alternative A, and m prefer alternative B, with $m = n - M < M$. We will use M and m to indicate both the sizes of the two groups and the groups' names. Each individual knows whether he belongs to M or to m. Individuals differ not only in the direction of their preferences, but also in their intensity. The model can incorporate different informational assumptions. For concreteness, we conduct our analysis in terms of two opposing parties, whose sizes and compositions are publicly known, while individual intensities of preferences are private information. In a later section, we discuss alternative assumptions.

The remainder of the model is borrowed from CLP. We summarize it briefly. Intensity is indicated by a value v_i representing the utility that individual i attaches to obtaining his preferred alternative, relative to the competing one: thus the utility experienced by i as a result of the committee's decision is v_i if i's preferred alternative is chosen, and 0 if it is not. We will use *intensity* and *value* interchangeably. Individual values are independent draws from a common and commonly known distribution $F(v)$ with support $[0, 1]$. We call $\mathbf{v}$ the vector of realized values.

Each individual is endowed with one indivisible vote. The group decision is taken through majority voting. Prior to voting, however, individuals can purchase or sell votes among themselves in exchange for a numeraire. The trade of a vote is an actual transfer of the vote and of all rights to its use. We normalize each voter's endowment of the numeraire to zero and allow borrowing at no cost. The important point is that no voter is budget constrained and all are treated equally.[8] Individual utility u_i is given by:

$$u_i = v_i I + t_i \tag{1}$$

where I equals 1 if i's preferred decision is chosen and 0 otherwise, and t_i is i's net monetary transfer, positive if i is a net seller of votes, or negative if he is a net buyer.

With two alternatives and a single voting decision, voting sincerely is always a weakly dominant strategy, and we restrict our attention to sincere voting equilibria: after trading, each individual casts all votes in his possession, if any, in support of the alternative he prefers. Our focus is on the vote trading mechanism, and specifically on a competitive spot market for votes.

We allow for probabilistic (mixed) demands. Let $S = \{s \in \mathbb{Z} \geq -1\}$ be the set of possible pure demands for each agent, where $\mathbb{Z}$ is the set of integers, and a negative demand corresponds to supply: agent i can offer to sell his vote, do nothing, or demand any positive integer number of votes. The set of strategies for each voter is the set of probability measures on S, ΔS, denoted by Σ. Elements of Σ are of the form $\sigma : S \to [0, 1]$ where, for each voter,

[8]Normalizing the endowment of the numeraire to zero and allowing borrowing simplifies the notation. As will become apparent, the analysis would be identical if each voter were granted an endowment of one unit of numeraire, with no borrowing.

$\sum_{s\in S}\sigma(s)=1$ and $\sigma(s)\geq 0$ for all $s\in S$.

If individuals adopt mixed strategies, the aggregate amounts of votes demanded and of votes offered need not coincide ex post. A rationing rule R maps the profile of voters' demands to a feasible allocation of votes. Indicating vectors by bold symbols, we denote the set of feasible vote allocations by $X=\{\mathbf{x}\in\mathbb{N}^n|\sum x_i=n\}$. The rule R is a function from realized demand profiles to the set of probability measures over vote allocations: $R: S^n\rightarrow\Delta X$. For all $\mathbf{s}\in S^n$, for any $\mathbf{x}$ in the support of $R(\mathbf{s})$, we require $x_i\in[\min(1,1+s_i),\max(1,1+s_i)]\ \forall i$, and $\mathbf{x}=\mathbf{1}+\mathbf{s}$ with probability 1 if $\sum s_i=0$. In words, no voter with positive demand can be required either to buy more votes than he demanded, or to sell his vote; no voter who offered his vote for sale can be required to buy votes, and all demands must be respected if they are all jointly feasible.

The particular mixed strategy profile, $\boldsymbol{\sigma}\in\Sigma^n$, and the rationing rule, R, imply a probability distribution over the set of final vote allocations that we denote as $r_{\boldsymbol{\sigma},R}(\cdot)$. For every possible allocation $\mathbf{x}\in X$, we denote by $\varphi_{i,x}$ the probability that the committee decision coincides with voter i's favorite alternative. Thus, given some strategy profile $\boldsymbol{\sigma}$, the rationing rule R, a vote price p, and equation (1), the ex ante expected utility of voter i is given by:

$$U_i(\boldsymbol{\sigma},R,p)=\sum_{\mathbf{x}\in X}r_{\boldsymbol{\sigma},R}(x)\left[\varphi_{i,x}v_i-(x_i-1)p\right] \tag{2}$$

Each individual makes his trading and voting choices so as to maximize (2).

2.1 The Definition of Equilibrium

To allow for the existence of mixed strategies, we must depart from requiring that realized demand always clear the market at the equilibrium price. The concept of *ex ante competitive equilibrium* substitutes the traditional requirement of market balance with the weaker condition that market demand and supply coincide *in expectation*. The discipline imposed by market equilibrium is softened to the requirement that deviations from market balance be unsystematic and unpredictable.

In equilibrium, individuals select strategies that maximize their expected utility, given the strategies used by others and the price. Demands are interdependent and best-respond to others' demands. In a market for votes, such interdependence is inevitable because the value of a vote depends on the full profile of vote allocations.[9]

When the two opposing groups have different sizes, the notion of ex ante equilibrium proposed in CLP needs to be extended. The reason is that best response strategies will generally differ across members of the two groups. As a result, even though demands are anonymous, the equilibrium, if it exists, conveys information about the direction of preferences associated to each demand, and individual strategies will take that information into account. In the spirit of rational expectations models, we call an equilibrium *fully revealing* if the price and individual strategies are identical to what they would be if all individual information were pooled–here, with full information.

Surveying the literature on the existence of rational expectations equilibria, Allen and Jordan (1998) identify the "competitive message"–the price and the set of others' demands–as the smallest possible information message that generically supports a fully revealing rational expectations equilibrium. The problem is that, in general, prices alone cannot reveal all information when the dimensionality of the price set is lower than the dimensionality of the state space. This is true in our environment, with a single price and a large-dimension state space–the realized values and the group each value belongs to. Thus we need to condition equilibrium behavior not only on the price but also on the set of others' demands. We say that the equilibrium is fully revealing if either: (1) the equilibrium price, together with the set of others' equilibrium demands and the rational belief that the market is in equilibrium, fully convey to voter i the direction of preferences associated to each demand; or (2) the information conveyed is partial but voter i has a unique best response, identical to his best response under full information.

Define $\sigma_i^*(\mathbf{v})$ as individual i's equilibrium strategy when all preferences are known, where $\mathbf{v}$ stands for the vector of realized intensity values. Then:

Definition. *The vector of strategies $\boldsymbol{\sigma}^*$ and the price p^* constitute a fully revealing ex ante competitive equilibrium relative to rationing rule R if the following conditions are satisfied:*

1. *For each agent i, σ_i^* satisfies*

$$\sigma_i^* \in \arg\underset{\sigma_i \in \Sigma}{Max} U_i\left(\sigma_i, \boldsymbol{\sigma}_{-i}^*, R, p^*\right)$$

[9] As a transparent example, all remaining votes have zero value if one voter holds a majority on his own. In competitive equilibrium theory, such interdependence is found in analyses of contributions to public goods (for example, Arrow and Hahn. 1971, pp.132-6).

2. In expectation, the market clears, i.e.,

$$\sum_{i=1}^{n} \sum_{\mathbf{s} \in S^n} \sigma_i^*(\mathbf{s}) \cdot \mathbf{s} = 0$$

3. Given $\{\boldsymbol{\sigma}_{-i}^*, p^*\}$ and the knowledge that the equilibrium is fully revealing,

$$\sigma_i^* = \sigma_i^*(\mathbf{v}) \text{ for all } i.$$

An important corollary is that if a fully revealing equilibrium exists, then it is also an equilibrium of the complete information game. Thus everything that follows applies identically to the alternative scenario in which all preferences and group memberships are commonly known. [10]

In general, the existence and the characterization of the equilibrium will depend on the rationing rule. In line with the anonymous, centralized trading of the competitive equilibrium model, we require that the rationing rule be anonymous: traders' orders are selected randomly and treated equally, independently of the group the trader belongs to, the intensity of his preferences, or the volume of his demand. Again following CLP, we concentrate here, and for most of the analysis, on a rule called $R1$ or Rationing-by-Voter. $R1$ requires that any positive demand for votes be either satisfied in full, or not at all: for any vector of realized demands $\mathbf{s}$, a final allocation $\mathbf{x}$ must satisfy $x_i \in \{1, 1 + s_i\}$ $\forall i$. Under $R1$, any outstanding positive order for votes is equally likely to be selected; the order is satisfied if there exists sufficient outstanding supply to do so fully, in which case the sellers are selected with equal probability among all voters with outstanding offers to sell. If the order cannot be fully satisfied, then it remains void. A second positive order is then randomly selected from those remaining, with equal probability, and the process continues until either all orders are satisfied or the only orders left outstanding are all infeasible. $R1$ is well-suited to a market for votes because the value of a package of votes can change discontinuously with changes of a single unit.[11] In the final section of the paper, we return to the rationing rule and discuss the conditions under which our results are robust to an alternative rule that

[10] Note that the reverse does not hold: an equilibrium of the full information game need not be a fully revealing equilibrium of the incomplete information game, because it may be impossible for an agent to extract all relevant information.

[11] $R1$ resembles All-or-Nothing (AON) orders used in securities trading: the order is executed at the specified price only if it can be executed in full.

allows for partially filled orders. Up to that point, all our results are to be read as relative to rationing rule $R1$**.**

An equilibrium with no trade always exists–if no-one else is trading, an individual is rationed with probability one–and is, trivially, fully revealing–strategies are identical to what they would be with full information. Our interest is in equilibria with trade.

If an equilibrium existed in pure strategies, market balance would hold not only ex ante but ex post, and no rationing would occur. We need to allow for mixed strategies and ex ante equilibrium because in a market for votes with two opposing groups of known sizes, no fully revealing competitive equilibrium with trade exists in pure strategies. This result is well-known[12] but we reproduce it here because it is the point of departure of our analysis.

Remark. *For all n odd, m, F, and $\mathbf{v}$, there is no price p^* and vector of strategies $\mathbf{s}^*(\mathbf{v}, p^*)$ such that $s_i^*(\mathbf{v}, p^*) = \arg\underset{s_i \in S}{Max} U_i\left(s_i, \mathbf{s}_{-i}^*, p^*\right)$ for all i and $\sum_i s_i^*(\mathbf{v}, p^*) = 0$, unless $s_i^*(\mathbf{v}, p^*) = 0$ for all i.*

Proof. The logic is simple. If there is trade, for all $p > 0$, $\sum_{i \in m} s_i^*(\mathbf{v}, p) \in \{-m, (M - m + 1)/2\}$: if the aggregate demand of minority voters is positive, it must equal the minimum number of votes required to win; alternatively, at any positive price all losing votes must be offered for sale. But $\sum_{i \in M} s_i^*(\mathbf{v}, p) \leq 0$: in equilibrium, the aggregate demand by majority voters cannot be positive. In addition, $\sum_{i \in M} s_i^*(\mathbf{v}, p) \neq -(M - m + 1)/2$: if $(M - m + 1)/2$ votes were traded, the remaining $(M + m - 1)/2$ votes collectively held by M voters would be worthless and thus offered for sale too. Thus for all $p > 0$, $\sum_{i \in m} s_i^*(\mathbf{v}, p) + \sum_{i \in M} s_i^*(\mathbf{v}, p) \neq 0$. If $p = 0$, $\sum_{i \in m} s_i^*(\mathbf{v}, p) \geq (M - m + 1)/2$,[13] but $\sum_{i \in M} s_i^*(\mathbf{v}, p) \geq -(M - m - 1)/2$, because the only supply can come from M voters whose vote is not pivotal. Thus for $p = 0$, $\sum_{i \in m} s_i^*(\mathbf{v}, p) + \sum_{i \in M} s_i^*(\mathbf{v}, p) > 0$. ∎

The question this paper addresses then is whether a fully revealing *ex ante* competitive equilibrium with trade exists, given the knowledge of m and M.

[12] Ferejohn (1974), Philipson and Snyder (1996), Piketty (1994), Kultti and Salonen (2005), Casella, Palfrey, Turban (2014)

[13] We are assuming that at $p = 0$, voters on the losing side demand rather than sell votes. This is equivalent to the standard assumption that goods are in excess demand at 0 price.

3 Equilibrium Existence and Characterization

In this section we derive two theorems. Theorem 1 identifies a sufficient condition guaranteeing that an ex ante equilibrium with trade exists and provides a chracterization of such an equilibrium. Theorem 2 shows that with large electorates the sufficient condition must be satisfied with probability arbitrarily close to 1.

Given realized values $\mathbf{v}$, we denote by $v_{(1)}$ the highest realized value; by $G \in \{m, M\}$ the group such that $v_{(1)} \in G$–the group to which the highest intensity individual belongs–, and by g the opposite group. We call $\overline{v}_G$ ($\overline{v}_g$) the highest realized value in G (g) (thus by definition $\overline{v}_G = v_{(1)}$).[14] Finally, we denote by $v_{(2)G}$ the second highest value in G: $v_{(2)G} = \max(v_i \in \{G \setminus \overline{v}_G\})$.

Theorem 1. *For all n odd, m, and F there exists a threshold $\mu(n) \in (0, 1)$ such that if $\overline{v}_g \geq \mu(n)v_{(2)G}$, there exists a fully revealing ex ante equilibrium with trade where $\overline{v}_G$ and $\overline{v}_g$ randomize between demanding $\frac{n-1}{2}$ votes (with probabilities $q_{\overline{G}} > 0$ and $q_{\overline{g}} > 0$ respectively) and selling their vote, and all other individuals sell.*

The theorem is proved in the Appendix. The expression for $\mu(n)$ is not particularly informative but it is worth noting that $\mu(n)$ is increasing in n for all $n > 3$ and converges to $1/2$ for large n.[15] The condition $\overline{v}_g \geq \mu(n)v_{(2)G}$ is necessary and sufficient for the existence of the equilibrium characterized in the theorem, and is thus sufficient for the existence of a fully revealing ex ante equilibrium with trade.[16] For clarity, recall that individual preferences are private information: $\bar{v}_G$ and $\bar{v}_g$'s group memberships as well as a voter's own position in the values' ranking–whether as $\bar{v}_G$ or $\bar{v}_g$, or not–are revealed in equilibrium.

The theorem says that if the condition is satisfied, an equilibrium exists that always assumes this form, regardless of the realized rankings in the values of the two groups. The equilibrium exists whether $G = m$ or $G = M$, and, because $\mu(n) < 1$ for all n, the equilibrium exists whether the two highest value voters are on opposite sides or on the same side, as long as $\overline{v}_g \geq \mu(n)v_{(2)G}$. The price p and the mixing probabilities, $q_{\overline{G}}$ and $q_{\overline{g}}$, depend on $\overline{v}_G$,

[14] Throughout the paper, we use v_i to denote the value of i but also occasionally, with abuse of notation, the name of voter i. We use the notation $v_{(1)}$ to indicate the highest draw, as opposed to the more standard $v_{(n)}$, for consistency with $v_{(2)G}$.

[15] If $n = 3$, $\mu = 2/3$.

[16] Theorem 1 does not state that no fully revealing equilibrium with trade exists if $\overline{v}_g < \mu(n)v_{(2)G}$. In a specific example ($M = 3$, $m = 2$), we have constructed such an equilibrium for value realizations that violate the condition (Casella, Palfrey and Turban, 2014).

$\overline{v}_g$, and on whether $G = m$, or $G = M$, but the structure of the equilibrium is unchanged: the highest-value individual belonging to M and the highest-value individual belonging to m compete for dictatorship, while all others sell their votes.

When it exists, the equilibrium recalls the equilibrium in CLP. In that paper's symmetric environment, the competition for dictatorship is between the two highest-value individuals overall; here it is between the two individuals with highest value *and* opposite preferences. The robustness of the result to the different assumptions in the two models highlights a central aspect of markets for votes. Note the similarity to the equilibrium of an all-pay auction: all potential bidders but the two with highest values abstain from the contest; the two with highest values submit strictly positive bids with positive probability, and the bids are anchored by the property that the expected payoff of the second highest value bidder is zero.[17] By softening the requirement of exact market clearing, the concept of ex ante equilibrium brings to light the essential nature of a market for votes: votes have no value in themselves, and a well-functioning market for votes approximates an auction for decision power. The market allocates such power to one of the two individuals with the highest incentive to compete for it.

In the scenario studied here, with two opposing groups of different sizes, the equilibrium has a number of additional features. First, there is a positive probability that the only realized purchases are made by $\overline{v}_M$, that is, by the majority. The result is less paradoxical than it seems: all other majority members are offering their votes for sale, and $\overline{v}_M$ buys to prevent the transfer of votes to the minority. Preemptive purchases by the majority are very plausible: any sponsor of a bill needs to worry about the support of his weakest allies. But to our knowledge they have no role in usual formalizations of vote trading. For the same reason, the equilibrium predicts intra-group trading with high probability for all m and M. Again, most voters are offering their vote for sale, and high value individuals need to preempt sales to the opposite group by their own weak allies.

Second, unless all of one's group votes are purchased, the winning majority will be larger than the minimal winning coalition. Thus in general the equilibrium predicts super-majority, a counter-intuitive result in a market for votes where votes command a positive price, the number of additional votes the minority needs to win is common knowledge, and all demands

[17] Baye et al., 1996; Hillman and Riley, 1989; Siegel, 2009. With two known opposite sides, M and m, the auction model would exhibit Identity-Dependent Externalities (Klose and Kovenock, 2013), with the result that the equilibrium exists only under some condition–as indeed we also find here.

are simultaneous.[18]

Characterizing the equilibrium mixing probabilities and the price provides a sharper picture of the equilibrium we have constructed. The condition for its existence is invariant to the group identity of the highest value individual (whether $G = M$, or $G = m$), and the notation in Theorem 1 makes that clear. However $q_{\overline{G}}$, $q_{\overline{g}}$ and p are not invariant and can be expressed more transparently if we account for group membership explicitly. Call $q_{\overline{m}}$ the probability with which $\overline{v}_m$ demands $(n-1)/2$ votes, and $1 - q_{\overline{m}}$ the probability with which $\overline{v}_m$ offers his vote for sale, and similarly for $q_{\overline{M}}$. Proposition 1 follows from the proof of Theorem 1 in the Appendix:

Proposition 1. *For all* $\mathbf{v}$ *such that the equilibrium in Theorem 1 exists, there exist two thresholds,* $\underline{\rho}(n)$ *and* $\overline{\rho}(n)$*, with* $1/2 \leq \underline{\rho}(n) < \overline{\rho}(n) < 1$ *for all* n*, such that:*

1. *If* $\overline{v}_m \geq \overline{\rho}(n)\overline{v}_M$*, then* $q_{\overline{m}} = 1$, $q_{\overline{M}} = \frac{n-1}{n+1}$*, and* $p = \frac{2}{n+1}\overline{v}_M$.
2. *If* $\overline{v}_m \in \left(\underline{\rho}(n)\overline{v}_M, \overline{\rho}(n)\overline{v}_M\right)$*, then* $q_{\overline{m}} \in \left(\frac{n-1}{n+1}, 1\right)$, $q_{\overline{M}} \in \left(\frac{n-1}{n+1}, 1\right)$*, and* $p \in \left(\frac{2}{n+1}\overline{v}_m, \frac{2}{n+1}\overline{v}_M\right)$.
3. *If* $\overline{v}_m \leq \underline{\rho}(n)\overline{v}_M$*, then* $q_{\overline{m}} = \frac{n-1}{n+1}$, $q_{\overline{M}} = 1$*, and* $p = \frac{2}{n+1}\overline{v}_m$.

If the equilibrium exists, given the size of the electorate n, $q_{\overline{m}}$, $q_{\overline{M}}$, and p depend exclusively on $\overline{v}_m$ and $\overline{v}_M$. If the disparity in the two values is large enough, then the voter with higher value always demands $(n-1)/2$ additional votes, while the other randomizes between such a demand and offering his vote for sale; if the two values are instead close, then both voters randomize.[19] Strikingly, neither the equilibrium strategies nor the price depend on the relative size of the two groups. The value of m affects the probabilities of the inter-party ranking in the realizations of values $\mathbf{v}$, but, given n and $\mathbf{v}$, if the equilibrium exists, $q_{\overline{m}}$, $q_{\overline{M}}$, and p are identical whether $m = 1$ or $m = M - 1$. The intuition is clear: since all individuals but $\overline{v}_m$ and $\overline{v}_M$ always offer their vote for sale, the precise numerical advantage of the majority is irrelevant in equilibrium. Either $\overline{v}_m$ too offers his vote for sale, and the majority wins, for any m; or $\overline{v}_m$ demands $(n-1)/2$ votes, and any demand by $\overline{v}_M$ lower then $(n-1)/2$ results in defeat with probability 1, for any m.

The details of the proposition tell us more. First, because $\underline{\rho}(n) < \overline{\rho}(n) < 1$, there exists $\overline{v}_m < \overline{v}_M$ such that $q_{\overline{m}} > q_{\overline{M}}$, but the reverse is never true. Thus not only do $\overline{v}_M$ and $\overline{v}_m$

[18] As remarked in the Introduction, a similar result in Groseclose and Snyder (1996) applies to a different model and has a different origin.

[19] The two thresholds $\underline{\rho}(n)$ and $\overline{\rho}(n)$ are defined precisely in the Appendix. Both converge to 1 at large n.

demand the same number of votes, if they demand votes at all, but the minority's strategy is always weakly more aggressive. It is not difficult to see why: if no trade is concluded, $\overline{v}_M$ is sure to win, while $\overline{v}_m$ is sure to lose; the less desirable outside option predisposes $\overline{v}_m$ towards buying. The same logic underlying the *underdog effect* in models of costly voting operates in a market for votes[20].

Second, although demanding a majority requires demanding more votes in larger electorates, the price of a vote decreases at rate n. Using the expressions for $\underline{\rho}(n)$ and $\overline{\rho}(n)$ in the Appendix, one can verify that a voter's expenditure, $p \cdot (\frac{n-1}{2})$ if not rationed, is always bounded above by $\min(\overline{v}_m, \overline{v}_M)$. Thus not only is the expenditure always finite, but, as must be true in equilibrium, individuals always gain from having their demand satisfied.

We can then complement the proposition with the required condition for equilibrium existence, in each of the three cases. Consider for example case (1) and suppose $n > 3$. Because $\mu(n) < 1/2 < \underline{\rho}(n) < \overline{\rho}(n) < 1$ for all $n > 3$, if $\overline{v}_m \in [\overline{\rho}(n)\overline{v}_M, \overline{v}_M]$, then $G = M$, and $\overline{v}_m > \mu(n)v_{(2)M}$ –hence if $\overline{v}_m \in [\overline{\rho}(n)\overline{v}_M, \overline{v}_M]$, the equilibrium exists. If $\overline{v}_m > \overline{v}_M$, then $G = m$, and existence of equilibrium requires the additional condition $\overline{v}_M > \mu(n)v_{(2)m}$. Hence $\Pr(\overline{v}_m \geq \overline{\rho}(n)\overline{v}_M, \overline{v}_M \geq \mu v_{(2)m})$ is the probability that the equilibrium exists and realized values falls under case (1). Using the same logic, no additional condition is required in case (2); case (3) requires $\overline{v}_m \geq \mu v_{(2)M}$. Thus, for all $n > 3$, in the equilibrium we have constructed:[21]

$$\begin{aligned}
\Pr\left(q_{\overline{m}} = 1\right) &= \Pr(\overline{v}_m \geq \overline{\rho}\overline{v}_M, \overline{v}_M \geq \mu v_{(2)m}) &\equiv& \Pr(\mathbf{v} \in A_m) \\
\Pr\left(q_{\overline{m}} \in \left(\frac{n-1}{n+1}, 1\right), q_{\overline{M}} \in \left(\frac{n-1}{n+1}, 1\right)\right) &= \Pr(\underline{\rho}\overline{v}_M < \overline{v}_m < \overline{\rho}\overline{v}_M) &\equiv& \Pr(\mathbf{v} \in A) \\
\Pr\left(q_{\overline{M}} = 1\right) &= \Pr(\overline{v}_m \leq \underline{\rho}\overline{v}_M, \overline{v}_m \geq \mu v_{(2)M}) &\equiv& \Pr(\mathbf{v} \in A_M),
\end{aligned}$$

where we use the notation A_m, A, A_M as short-hands for the corresponding regions of the realizations of $\mathbf{v}$. The subscript identifies the group of the voter who demands with probability 1. By Proposition 1, no other value realizations can support the equilibrium of Theorem 1, and no other strategies can be observed in such an equilibrium.

As n increases, the likelihood of realizations for which the equilibrium does not exist falls drastically. For instance, if F is uniform and $\alpha = \frac{1}{3}$, the equilibrium exists with probability

[20] Simon (1954), Gartner (1976); Levine and Palfrey (2007)

[21] Because $\mu(3) = 2/3 > 1/2$, the case $n = 3$ is slightly different (see the Appendix). We treat it separately in our proofs but to avoid clutter do not discuss it in the text.

98.3% with $n = 9$, and more than 99.9% with $n = 21$. More generally, if the minority is a non-vanishing fraction of the electorate,[22] then with independent draws from any common distribution F, at large n, both $\overline{v}_g/\overline{v}_G$ and $v_{(2)G}/\overline{v}_G$ must approach the upper bound of the distribution's support. It then follows that when the electorate is large, the restriction on realized values required for the existence of the equilibrium described in Theorem 1 is almost certainly satisfied. Indeed this is our second result. Suppose $m = \lfloor \alpha n \rfloor$ for all n, where $\lfloor \alpha n \rfloor$ is the largest integer not greater than αn, and α is a constant in $(0, 1/2)$. Adding a subscript n to indicate explicitly the dependence on the size of the market, we can state:

Theorem 2. *Consider a sequence of vote markets. For any* $\alpha \in \left(0, \frac{1}{2}\right)$ *and* F, $\lim_{n\to\infty} \Pr{}_n(\overline{v}_{g,n} \geq \mu(n) v_{(2)G,n}) = 1$.

Proof. The proof of the theorem is immediate. Given $\mu(n) < 1/2$, the theorem follows if $\lim_{n\to\infty} \Pr_n(\overline{v}_{g,n} > 1/2) = 1$. But $\lim_{n\to\infty} \Pr_n(\overline{v}_{g,n} > 1/2) = \lim_{n\to\infty} 1 - [F(1/2)]^{\lfloor \alpha n \rfloor} = 1$, and the result is established. ∎

Theorem 2 implies that for large n the equilibrium described in Theorem 1 exists with probability that approaches 1. In addition, because in such an equilibrium the probabilities with which $\overline{v}_G$ *and* $\overline{v}_g$ demand $(n-1)/2$ votes are bounded below by $(n-1)/(n+1)$, at large n both probabilities must also approach 1. Theorem 2 thus leads to the following Corollary:

Corollary 1. *For any* $\alpha \in \left(0, \frac{1}{2}\right)$ *and* F, $\Pr[\lim_{n\to\infty} q_{\overline{G},n}(\mathbf{v}) = 1] = 1$, *and* $\Pr[\lim_{n\to\infty} q_{\overline{G},n}(v) = 1] = 1$

4 Market Outcomes

4.1 Frequency of minority victories

The most unexpected feature of Theorem 1 is that when the equilibrium exists the market outcome depends on the size of the minority only indirectly. As we remarked, if the equilibrium exists, given realized values the expected outcome is the same whether there is a single minority voter or the minority comprises almost half of the electorate. Together with the weakly more aggressive minority strategy highlighted by Proposition 1, this result suggests

[22] I.e. $\frac{m}{n}$ is bounded away from 0 as $n \longrightarrow \infty$.

a systematic vote market bias in favor of the minority group: a higher frequency of minority victories than efficiency dictates.

To evaluate this conjecture, we need to construct an equilibrium that exists for all value draws, and define an efficiency benchmark. Since an equilibrium with no trade exists trivially for all value realizations, we can construct an equilibrium such that if $\overline{v}_g \geq \mu(n)v_{(2)G}$, then trade occurs and the equilibrium of Theorem 1 is selected; if $\overline{v}_g < \mu(n)v_{(2)G}$, then no vote-trading takes place and the majority wins with probability 1. Our equilibrium construction thus minimizes the frequency of minority victories when the condition is not met.[23] We call θ_m the ex ante expected frequency of minority victories in such an equilibrium, before values are drawn. Recall that $x(\mathbf{v})$ is a random variable denoting a final allocation of votes for a given value profile. Hence: $\theta_m \equiv \Pr_F(\sum_{i\in m} x_i(\mathbf{v}) > \sum_{j\in M} x_j(\mathbf{v}))$.

In line with the anonymity of the competitive market and of majority voting, we measure efficiency by ex ante efficiency, treating each voter identically–expected utility before the voter knows the group he belongs to and before values are drawn. Ex ante efficiency is equivalent to the utilitarian criterion: it is maximized when, for each realization of values, the group with higher aggregate value prevails. We call θ_m^* the expected frequency of minority victories under this efficiency benchmark: $\theta_m^* \equiv \Pr_F(\sum_{i\in m} v_i > \sum_{j\in M} v_i)$. To evaluate whether a systematic pro-minority bias is indeed realized, in this section we compare θ_m to θ_m^*.

We begin by establishing a preliminary result. Because it can be of some general interest, we report it here as a separate lemma.

Lemma 1. *For all distributions F, if all v_i, $i \in m$ and $i \in M$ are i.i.d. with distribution F, then for all n and m, $\theta_m^* \leq \frac{1}{1+\binom{M}{m}} \leq \frac{m}{n}$.*

The lemma is proved in the Appendix. It states that if values are i.i.d., then for any distribution F the expected share of value configurations such that the aggregate minority value is larger than the aggregate majority value, and thus a minority victory is efficient, cannot be larger than the share of the minority in the electorate. The statement is intuitive because it establishes an upper bound for θ_m^* that holds for all F, n, and m and can be compared to θ_m, the equilibrium fraction of expected minority victories.

Conditional on value realizations, $\theta_m(\mathbf{v})$ is either characterized precisely by the strategies in Theorem 1, or equals 0, by our equilibrium construction, if the condition in Theorem 1

[23] As noted earlier, equilibria with trade may exist when $\overline{v}_g < \mu(n)v_{(2)G}$, and thus the expected fraction of minority victories must be weakly higher than in our equilibrium construction.

is not satisfied. In particular, because under Theorem 1 the final votes' allocation depends only on the probability with which $v_{\overline{m}}$ and $v_{\overline{M}}$ demand votes, we can write:

$$\theta_m(\mathbf{v}) = \begin{cases} q_{\overline{m}}(\mathbf{v})(1 - q_{\overline{M}}(\mathbf{v})) + \frac{1}{2} \cdot q_{\overline{m}}(\mathbf{v}) q_{\overline{M}}(\mathbf{v}) & \text{if } \overline{v}_g \geq \mu(n) v_{(2)G} \\ 0 & \text{if } \overline{v}_g < \mu(n) v_{(2)G} \end{cases}$$

where the notation recognizes explicitly that the equilibrium buying probabilities $q_{\overline{m}}$ and $q_{\overline{M}}$ depend on the realized values $\mathbf{v}$. Using the values of – or bounds on – $q_{\overline{m}}$ and $q_{\overline{M}}$ described in Proposition 1, we can thus find a lower bound for θ_m:

$$\theta_m \geq \left(\frac{n+3}{2(n+1)} \right) \Pr(\mathbf{v} \in A_m) + \left(\frac{n-1}{2(n+1)} \right) [\Pr(\mathbf{v} \in A_M) + \Pr(\mathbf{v} \in A)] \equiv \underline{\theta_m} \quad (3)$$

with strong inequality if $\Pr(\mathbf{v} \in A) > 0$. The probability of realizations in the different regions of the value space depends on F, and thus so does θ_m. Yet, as we prove in the Appendix:

Proposition 2. *For all n, m , and F, $\theta_m > \theta_m^*$.*

Relatively to utilitarian efficiency, the market, at least in the equilibrium we have characterized, *always* leads to excessive minority victories. Remarkably, the conclusion holds for all electorate sizes, regardless of the size of the minority and of the shape of the values distribution. An example can help in making the proposition concrete. Suppose that F is uniform. Figure 1 plots $\underline{\theta_m}$, on the vertical axis, against $m/n \equiv \alpha$ on the horizontal axis, with $m = 1, .., (n-1)/2$. The different panels correspond to different values of n: $n = 9$, 15, and 21. In each panel, the 45^0 line thus equals $m/n = \alpha$, and by Lemma 1, since $\theta_m^* \leq m/n$, if $\underline{\theta_m} > m/n$, it follows that $\theta_m > \theta_m^*$. The figure shows that $\underline{\theta_m}$ can be surprisingly large, especially at low m/n. For example, if $m = 1$, $\underline{\theta_m}$ is 33 percent at $n = 9$ (when m is 11 percent of the voters) and remains almost 29 percent at $n = 21$ (when m is just below 5 percent of the voters).

In a large electorate, the expected fraction of equilibrium minority victories can be made precise. The result confirms the magnitude of the pro-minority bias at low m/n highlighted by Figure 1. The points of departure are Theorem 2 and its Corollary in the previous section: if n is large, with probability approaching 1, realized values satisfy the condition in Theorem 1, and again with probability approaching 1, voters $\overline{v}_m$ and $\overline{v}_M$ both demand $(n-1)/2$ votes, while all other voters offer their votes for sale. An immediate and unexpected result then

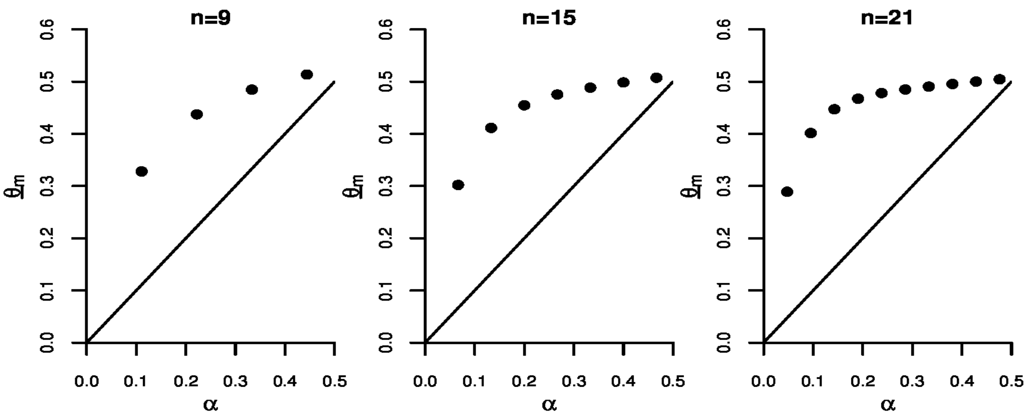

Figure 1: Lower bound on the probability of minority victories, as function of $\alpha = \frac{m}{n}$. Values are i.i.d. drawn from a uniform distribution.

follows: the final outcome depends exclusively on which one of $\overline{v}_m$ and $\overline{v}_M$ has his order filled, and since both have identical chances, both win with equal probability. Theorem 2 and its Corollary directly imply: [24]

Proposition 3. *Consider a sequence of vote markets, such that for all n, $m = \lfloor \alpha n \rfloor$, with $\alpha \in \left(0, \frac{1}{2}\right)$. Then for any α and F, $\lim\limits_{n\to\infty} \theta_{m,n} = \frac{1}{2}$. Moreover,* $\Pr[\lim\limits_{n\to\infty} \theta_{m,n}(\mathbf{v}) = \frac{1}{2}] = 1$.

At sufficiently large market size, the minority is expected to win with probability arbitrarily close to 1/2, for *any* minority share and for *any* distribution from which values are drawn. Note that the proposition is very strong; it states not only that the ex ante expected frequency of minority victories (θ_m) converges to 1/2, but that the expected frequency of minority victories converges to 1/2 for all value realizations, except on a set with zero probability ($\theta_m(\mathbf{v}) \underset{a.s}{\longrightarrow} 1/2$).

Given the previous results, the intuition is straightforward, but the result remains surprising. Whether the minority is 40 percent of the total electorate, 25 percent, or 10 percent, as long as it is not negligible, in a sufficiently large vote market there is an equilibrium such that the minority wins with probability 1/2 for any shape of the value distribution. After trade, the minority and the majority group are *equally likely* to control a majority of the votes. The market nullifies majority voting: following the will of the electorate becomes identical to flipping a coin.

[24] For $\mathbf{v}$ satisfying the condition in Theorem 1, $\theta_{m,n}(\mathbf{v})$ is a continuous function of $q_{\bar{G},n}(\mathbf{v})$ and $q_{\bar{g},n}(\mathbf{v})$. By Theorem 2, its Corollary, and the continuous mapping theorem, $\theta_{m,n}(\mathbf{v}) \underset{a.s}{\longrightarrow} 1/2$.

Although Proposition 3 is a limit result, Figure 1 shows that convergence towards high fractions of minority victories can be very fast. We noted that, with F uniform, at $n = 9$, $m = 1$ corresponds to $\underline{\theta_m} = 0.33$. Holding $m = \lfloor n/9 \rfloor$, $\underline{\theta_m}$ grows quickly with n: it is already higher than 0.40 at $n = 19$ (with only two voters belonging to the minority), and just below 0.45 at $n = 27$. The limit case is particularly stark but the logic it highlights underlies the small sample results as well.

4.2 Welfare

Beyond the existence of a bias, we are finally interested in the welfare properties of the market. Since $\theta_m > \theta_m^*$, we know that the market falls short of efficiency. But how does the market compare to majority voting in the absence of vote trading? To address this question we need a direct comparison of ex ante utilities. We call W the ex ante expected utility in the equilibrium we have constructed, and W_0 the ex ante expected utility in the absence of vote trading (i.e. with simple majority voting), and denote $\mathcal{A} = A_M \cup A_m \cup A$:

$$nW \;=\; \int_{\mathbf{v}\in\mathcal{A}} \left[(1-\theta_m(\mathbf{v}))\sum_{i\in M} v_i + \theta_m(\mathbf{v})\sum_{j\in m} v_j\right] dF^n(\mathbf{v}) + \int_{\mathbf{v}\notin\mathcal{A}} \left[\sum_{i\in M} v_i\right] dF^n(\mathbf{v}) \quad (4)$$

$$nW_0 = \int_{\mathbf{v}} \left[\sum_{i\in M} v_i\right] dF^n(\mathbf{v}) \quad (5)$$

If n is small, the welfare comparison between the vote market and no-trade depends on the shape of the value distribution. Define $E_F(v)$ and $E_{F,n}(v_{(1)})$ as, respectively, the expectation of a random variable with distribution F, and the expectation of the maximum of an i.i.d. sample of size n drawn from F.[25] Then:

Proposition 4. *For all F, n, and m, if $E_F(v) > \frac{2(n-m)}{n(n-2m+1)} E_{F,n}(v_{(1)})$ then $W < W_0$.*

Proposition 4 states that if F's expected value is not too small relative to the expected highest order statistics, then expected welfare with the market must be lower than expected welfare with simple majority voting and no trade. Note that the condition it establishes is sufficient, not necessary. Predictably, the condition is more stringent–the market's relative performance improves–the larger is the minority. If we set $m = \frac{n-1}{2}$, the condition simplifies to $E_F(v) > \left(\frac{n+1}{2n}\right) E_{F,n}(v_{(1)})$, a stricter requirement guaranteeing $W < W_0$ for all m and n.

[25] We use the notation $E_{F,n}(v_{(1)})$, as opposed to $E_F(v_{(n)})$, for consistency with $v_{(1)}$.

Why is the ratio $Ev/Ev_{(1)}$ important?[26] In the best of cases, i.e. when $\overline{v}_m = v_{(1)}$, a minority victory means that the smaller side prevails, reflecting the realization among its members of a particularly high value. The higher is the ratio $Ev/Ev_{(1)}$, the smaller is the expected distance between the high outlier and the other realized values, and the more costly is the neglect of the majority's larger size. Suppose for example that $F = v^b$, with $b > 0$. Then $Ev/Ev_{(1)} = (bn+1)/(bn+n)$, a ratio increasing in b. The condition in the proposition is satisfied for all $b > 1/(n-2m)$.[27]

The complications tied to the specific shape of F disappear when the market is large. Setting $m = \lfloor \alpha n \rfloor$, the condition in Proposition 4 becomes:

$$E_F(v) > \frac{2(1-\alpha)}{n(1-2\alpha)+1} E_{F,n}(v_{(1)}) \Longrightarrow W_n < W_{0n}.$$

For any $\alpha < 1/2$, $\lim_{n->\infty} \frac{2(1-\alpha)}{n(1-2\alpha)+1} = 0$: at very large n the condition is always satisfied, for any F. As in the previous asymptotic results, the finding can be stated in stronger terms: the welfare ranking holds not only in expected terms but as almost sure convergence; that is, in the limit, for all realizations of values, except a zero probability set:[28]

Proposition 5. *Consider a sequence of vote markets. For any* $\alpha \in (0, \frac{1}{2})$ *and* F, $\lim_{n \longrightarrow \infty}(W_n/W_{0n}) = \frac{1}{2(1-\alpha)} < 1$. *Moreover:* $\Pr[\lim\limits_{n \longrightarrow \infty} \frac{W_n(\mathbf{v})}{W_{0n}(\mathbf{v})} = \frac{1}{2(1-\alpha)}] = 1$.

For any non-trivial minority size and for any distribution of values, with a sufficiently large electorate vote-trading lowers welfare. Note the contribution of the proposition. The

[26] We omit the subscript F, trusting that there will be no confusion.

[27] Given a specific F, tighter conditions can be found. We show in Casella and Turban (2012) that if $F = v^b$, then $EW < EW_0$ if $b \geq 1$, a tighter bound which includes the uniform distribution.

[28] For all $\mathbf{v}$ that satisfy the condition in Theorem 1:

$$nW_n(\mathbf{v}) = \left[(1 - \theta_{m,n}(\mathbf{v})) \sum_{i \in M} v_i + \theta_{m,n}(\mathbf{v}) \sum_{j \in m} v_j \right]$$

In addition for any such $\mathbf{v}$, by Theorem 1, $\theta_{m,n}(\mathbf{v}) \in \left[\frac{n-1}{2(n+1)}, \frac{n+3}{2(n+1)}\right]$. Thus, for such values:

$$nW_n(\mathbf{v}) \in \left[\frac{n-1}{2(n+1)} \sum_{i=1}^{n} v_i, \frac{n+3}{2(n+1)} \sum_{i=1}^{n} v_i \right]$$

Theorem 2, the continuous mapping theorem, and the strong law of large numbers then give us immediately $W_n(\mathbf{v}) \underset{a.s}{\longrightarrow} Ev/2$. But by (5) and the strong law of large numbers, $W_{0,n}(\mathbf{v}) \underset{a.s}{\longrightarrow} (1-\alpha)Ev$. Using the continuous mapping theorem a final time, we then obtain $(W_n(\mathbf{v})/W_{0,n}(\mathbf{v})) \underset{a.s}{\longrightarrow} \frac{1}{2(1-\alpha)}$.

assumption of i.i.d. value draws implies that, in terms of ex ante expected utility, majority voting without trade must be asymptotically efficient. But there is no a priori reason why a market for votes should not be. If the price becomes negligible (as the probability that a single vote be pivotal becomes negligible), a market for votes could in principle support an equilibrium with negligible minority victories, and negligible efficiency losses. By Proposition 3, however, we know that this is not the case: the minority is always expected to win as frequently as the majority wins. As a result, the efficiency loss is both precisely quantifiable and significant. If the minority is a third of the electorate, for example, the loss in ex ante utility is 25 percent; if it is 15 percent, the loss is more than 40 percent.

5 Robustness of the equilibrium

5.1 Alternative information assumptions

We described the model in Section 2 by stating that both the precise values of m and M and the compositions of the two group are commonly known. As mentioned earlier, however, our results extend to a range of different informational scenarios.

Knowing the exact composition of each group–which voter belongs to which group–plays no role in the analysis because demands in the competitive market are anonymous. The assumption that group membership is known seems preferable in the case of small committees and inappropriate in the case of a large electorate. Both scenarios are consistent with the equilibrium we have characterized.

Exact knowledge of the sizes of the two groups is not required either. Theorem 1 relies on one central assumption: each voter knows that a majority and a minority exist and knows which group he belongs to. Given this, the proof does not depend on precise information on the values of m and M. In particular, equilibrium strategies do not require individuals to form expectations of the two group sizes. Intuitively, the exact sizes are irrelevant because in equilibrium, for *any* m and M, the only two demands with positive probability correspond to $(n-1)/2$ votes, while everyone else sells. The results on the expected frequency of minority victories and on ex ante expected utility also hold unchanged if there is uncertainty about group sizes: because they hold for any m and M, they hold when the sizes are uncertain[29].

[29]In the case of large electorate, suppose $m = \lfloor \alpha n \rfloor$, where α is a random variable distributed according to some CDF H over $[a, b]$ with $a > 0$, $b < \frac{1}{2}$. For the proof of Theorem 2, note that $P(\bar{v}_g \geq \mu(n)v_{(2)G}) \geq P(\bar{v}_g \geq \frac{1}{2}) \geq 1 - [F(1/2)]^{\lfloor an \rfloor}$. For $a > 0, lim_{n\to\infty} 1 - [F(1/2)]^{\lfloor an \rfloor} = 1$. The result follows. For Propositions

The results are robust not only to introducing more uncertainty, but also to the opposite change in assumptions: introducing more information. Because of the focus on a fully revealing equilibrium, the analysis remains identical if we assume that demands are *not* anonymous, so that not only m and M and the groups' composition are publicly known, but so is the identity of the voter expressing each demand. In fact, as noted previously, full revelation in equilibrium means that the results remain identical when all voters' preferences are public information – not only the direction of preferences but also the realized value for each voter. The assumption of complete information would allows us to substitute knowledge of the full competitive message with rational beliefs about strategies. Influential voices in the literature defend knowledge of both the equilibrium price and demands as the correct understanding of competitive equilibrium (Hurwicz (1977) and Mount and Reiter (1974)). It remains, however, a high informational requirement. In some situations, especially with small committees and members who interact regularly, full information on preferences may be a better modeling alternative.

5.2 An alternative rationing rule

The equilibrium strategies appear extreme: individuals either demand a majority of votes or sell. As in CLP, we want to verify that this is not an artefact of the all-or-nothing rationing rule (either an order is fully filled or it is passed over). CLP consider the following alternative rule, which they call $R2$, or rationing-by-vote. If voters' orders result in excess supply, the votes to be sold are chosen randomly from each seller, with equal probability. If instead there is excess demand, any vote supplied is randomly allocated to one of the individuals with outstanding purchasing orders, with equal probability. An order remains outstanding until it has been completely filled. When all supply is allocated, each individual who put in an order must purchase all units that have been directed to him, even if the order is only partially filled. Formally, we require $x_i \in \{0, 1, 2, .., 1+s_i\}$ for any x in the support of $R2\,(s)$. Like $R1$, $R2$ is anonymous. Contrary to $R1$, it guarantees that only one side of the market

2 and 4, we have not verified whether almost sure convergence holds when α is uncertain, but the results on expectations extend immediately. For Proposition 3, denote $\theta_{m,n}(\alpha)$ the expected fraction of minority victories, given α. Hence $\theta_{m,n} = \int_a^b \theta_{m,n}(\alpha)dH(\alpha)$. For all α, $\theta m_{,n}\,(\alpha) \to \frac{1}{2}$. In addition, for all n, α, $|\theta_{m,n}(\alpha)| < 1$. Hence by the bounded convergence theorem, $\theta_{m,n} \to \int_a^b \frac{1}{2} dH(\alpha) = \frac{1}{2}$. Identical reasoning can be used for Proposition 5. For any given α, denote $W_n(\alpha)$ the equilibrium welfare. Thus $W_n = \int_a^b W_n(\alpha)dH(\alpha)$. For all α, $W_n(\alpha) \to \frac{Ev}{2}$, and for all n, α, $|W_n(\alpha)| < 1$. By the bounded convergence theorem, $W_n \to \int_a^b \frac{Ev}{2} dH(\alpha) = \frac{Ev}{2}$. We can proceed likewise for W_0.

is ever rationed, but its requirement that partially filled orders be accepted seems ill-suited to a market for votes, where the value of votes hinges on pivotality, and thus on the exact number of votes transacted.

At $n = 3$, $R2$ and $R1$ are identical and Theorem 1 applies. Suppose then $n > 3$:

Theorem 3. *Suppose $R2$ is the rationing rule. For all n odd, m, and F there exists a threshold $\mu(n) \in (0, 1)$ such that if $\overline{v}_g \geq \mu_{R2}(n) Max[v_{(2)G}, v_{(2)g}]$, there exists a fully revealing ex ante equilibrium with trade where $\overline{v}_G$ and $\overline{v}_g$ randomize between demanding $(n-1)/2$ votes (with probabilities $q'_{\overline{G}} > 0$ and $q'_{\overline{g}} > 0$ respectively) and selling their vote, and all other individuals sell.*

The theorem is proved in Appendix B. Its similarity to Theorem 1 is apparent. There are two main differences: first, the thresholds in the two theorems differ, and $\mu_{R2}(n) > \mu(n)$, implying that the equilibrium exists under $R2$ under more restrictive conditions than under $R1$. In particular, $\lim_{n \longrightarrow \infty} \mu_{R2}(n) = \infty$: whereas under $R1$ the probability that the equilibrium exists in a very large market converges to 1, the probability converges to 0 under $R2$. Second, as can be verified in the Appendix, when the equilibrium exists, the equilibrium price p' is consistently lower than p, the equilibrium price under $R1$. The intuition is clear: when both $\overline{v}_G$ and $\overline{v}_g$ submit demands for $(n-1)/2$ votes, one of the two will receive and be charged for $(n-3)/2$ votes, useless votes, since the opponent will hold a majority. To compensate for this risk, the equilibrium price must be lower.[30]

The choice of rationing rule poses a number of interesting but challenging questions. We know that in general the equilibrium must depend on the exact rule, and we can debate whether the rationing rule is better thought of as part of the institution, controlled by the market designer, or as part of the equilibrium, and interpreted as reduced form for the complex, decentralized system of search that underlies the trades.[31] Our goal here is not to address these broad questions but to make a narrower point: Theorem 3 shows that the equilibrium discussed in this paper is not the artefact of one specific rationing rule, and in particular of the all-or-nothing nature of $R1$.

[30] There is a third difference as well. As the proof in Appendix B makes clear, the condition $\overline{v}_g \geq \mu_{R2}(n) Max[v_{(2)G}, v_{(2)g}]$ is sufficient for the existence of the equilibrium in Theorem 3–there are value realizations for which weaker conditions are necessary–whereas under $R1$ the condition in Theorem 1 is necessary and sufficient for the equilibrium characterized there.

[31] See for example Green (1980) for a compelling exposition of the second interpretation.

5.3 Correlated and not identically distributed values

We have assumed so far that values are independent both across groups and within groups, and identically distributed according to some distribution F. The assumption allowed us to provide simple closed form solutions, but the logic of the arguments shows that neither independence nor a common distribution are necessary for our more substantive results. Theorem 1 states a sufficient condition for a trading equilibrium that depends only on the existence of a sufficient wedge between $\overline{v}_g$ and $\overline{v}_{(2)G}$, the *realized* highest values in the two groups. Nor does the equilibrium depend on F: given m, M, R, p, and others' strategies, a voter's best response is fully identified. The probability that the condition in Theorem 1 is satisfied does depend on F, but the asymptotic result in Theorem 2 is robust to significant generalization.

Particularly relevant to our voting environment is the possibility of correlation in values. Consider then the following standard model, where the assumption of independence is weakened to conditional independence:

$$\begin{aligned} v_i &= v_m + \varepsilon_i \text{ for all } i \in m \\ v_j &= v_M + u_j \text{ for all } j \in M \end{aligned}$$

where v_m (v_M) is a common value shared by all m (M) voters, and ε_i and u_j are idiosyncratic components, independently drawn from distribution $G_m(\varepsilon)$, with full support $[0, \overline{\varepsilon}]$, and $G_M(u)$, with full support $[0, \overline{u}]$. For all fixed $\alpha \in (0, 1/2)$, as $n \longrightarrow \infty$, $\overline{v}_m \longrightarrow v_m + \overline{\varepsilon}$, and $\overline{v}_M \longrightarrow v_M + \overline{u}$. Thus for all $2(v_M + \overline{u}) \geq (v_m + \overline{\varepsilon}) \geq \frac{v_M + \overline{u}}{2}$ the equilibrium of Theorem 1 exists with probability approaching 1 asymptotically.[32] And if the equilibrium exists, Proposition 3 follows: asymptotically, the minority is expected to win with probability 1/2.

Relative to our previous results, there are then two qualifications. First, to ensure that the equilibrium always exists asymptotically, we need additional conditions on the distributions of values, here on v_m, $\overline{v}_M$, $\overline{\varepsilon}$, and $\overline{u}$. Second, the welfare results need to be re-evaluated and again in general will depend on the distributions. In this example, if $v_m + E_{G_m}(\varepsilon)$ is sufficiently larger than $v_M + E_{G_M}(u)$, then, depending on α, the vote market could be asymptotically superior to simple majority voting. If the distributions differ between the two groups, predictably the conclusions will depend on *how* they differ. Note however that neither qualification stems from relaxing independence. Our asymptotic results require that

[32] We are using $\lim_{n \longrightarrow \infty} \mu(n) = 1/2$.

the extremum statistic of the value draws in each group should converge to the upper bound of the support. The condition is violated if all values are perfectly correlated, but can accommodate high degrees of dependence[33].

6 Conclusions

How does a vote market function when voters are aware of their minority and majority status? In this paper, we have borrowed the concept of ex ante competitive equilibrium from Casella, Llorente-Saguer and Palfrey (2012) (CLP) and extended their model to an asymmetric setting where each voter knows that a majority and a minority exist, and knows which of the two groups he belongs to. We have characterized a sufficient condition for the existence of an ex ante equilibrium with trade for any electorate size, any majority advantage, and any distribution of intensities. In equilibrium, only two voters, the highest intensity voters on each side, demand votes with positive probabilities; all others offer their votes for sale. The two voters assign positive probability to only two actions: either selling, or demanding enough votes to alone control a majority of all votes.

The similarity to the equilibrium in CLP, where individuals are symmetric and equally likely to favor either alternative, suggests to us that, by re-establishing existence, the concept of ex ante equilibrium sheds light on a fundamental aspect of vote markets: votes per se are worthless; what is traded is decision power. The market becomes an auction for power. In line with well-known results from all-pay auctions, only the two individuals who most value the ownership of such power compete for it.

The probability of either group's victory depends only on the action of its most intense member and gives no direct weight to the size of the group. For any number of voters, any minority size, and any distribution of intensities, the market results in more frequent minority victories than efficiency dictates. In a large electorate, strikingly, the minority always wins with probability one half, regardless of its relative size. The systematic bias in favor of the minority exacts welfare costs, and the market can be welfare inferior to simple

[33]For example, statisticians working on limit distributions for maxima have proposed the concept of m-*dependence*. When values are drawn in a natural sequence (think of floods over time), m-dependence applies when there exists a finite m such that draws that are more than m steps apart are independent (Hoeffding and Robbins, 1948). In our application, the concept could be relevant for geographically or ideologically concentrated subgroups of voters. Theorem 2 and Proposition 5 continue to hold in this case, under minor regularity assumptions.

majority voting with no vote trading.

The results we have obtained are surprisingly clear-cut for such a long-debated problem. They depend on the concept of ex ante equilibrium and the implementation of such an equilibrium in our model–more precisely on the rationing rule. We focus our research project on competitive equilibrium, because we consider it the first tool of an economist and the first line of analysis of a trading problem. Thus we have restricted all trades to take place via a single price, in centralized, anonymous exchanges. In line with this approach, we have studied two rationing rules, both anonymous. The two rules support equilibria that are very closely related, in fact have identical structure, strengthening our beliefs that, given competitive equilibrium, the results depend on the special nature of the good traded–votes–more than on the details of the model. More support for this conclusion comes from experimental results. Both CLP and Casella, Palfrey and Turban (2014) test the model though the traditional platform used in market experiments: a continuous double auction with publicly visible bids and asks. The platform does not impose any rationing rule or any aggregate market clearing requirement, letting subjects propose and accept offers as they see fit within a specified time limit. The experimental results are in line with the theoretical predictions: the average transacted price converges towards the equilibrium price, purchases of votes are heavily weighted towards the two highest-intensity voters, whether overall in the symmetric case (CLP), or in each of the two groups, in the presence of asymmetry (Casella, Palfrey and Turban). When the two groups have different sizes, the inefficiently high fraction of minority victories is confirmed experimentally in every single session, and so is the welfare loss, relative to no trade. At least for these experiments, the conceptual model we have built, based on the notion of ex-ante competitive equilibrium and an anonymous rationing rule, appears to have predictive power.

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A Proofs

A.1 Proof of Theorem 1.

Theorem 1. *For all n odd, m, and F there exists a threshold $\mu(n) \in (0,1)$ such that if $\overline{v}_g \geq \mu(n)v_{(2)G}$, there exists a fully revealing ex ante equilibrium with trade where $\overline{v}_G$ and $\overline{v}_g$ randomize between demanding $\frac{n-1}{2}$ votes (with probabilities $q_{\overline{G}} > 0$ and $q_{\overline{g}} > 0$ respectively) and selling their vote, and all other individuals sell.*

Proof. The threshold $\mu(n)$ is given by:

$$\mu(n) = \begin{cases} \frac{2}{3} & \text{if } n = 3 \\ \max\left\{\frac{(n-2)(n-1)}{2(n^2+n-5)}, \frac{(n-2)(n-1)(n+1)}{2(n^3+3n^2-19n+21)}\right\} & \text{if } n > 3 \end{cases} \tag{6}$$

The theorem is implied by the following two lemmas. Lemma 2 characterizes the case $G = M$ and Lemma 3 the case $G = m$.

Lemma 2. *Suppose $G = M$ (or $\overline{v}_G = \overline{v}_M$ $\overline{v}_g = \overline{v}_{m,}$). Then if $\overline{v}_m \in \left[\mu(n)v_{(2)M}, \overline{v}_M\right]$, the strategies described in the theorem are a fully revealing ex ante competitive equilibrium for all n odd, m, and F. The mixing probabilities $q_{\overline{M}}$ and $q_{\overline{m}}$ and the price p depend on the realizations of $\overline{v}_m$ and $\overline{v}_M$. There exist two thresholds $\frac{1}{2} \leq \underline{\rho}(n) < \overline{\rho}(n) < 1$ such that:*

(a) Case $n > 3$

1. *If $\overline{v}_m \in \left[\mu(n)v_{(2)M}, \underline{\rho}(n)\overline{v}_M\right]$, $q_{\overline{M}}$, $q_{\overline{m}}$, and p satisfy:*

$$\begin{aligned} q_{\overline{M}} &= 1 \\ q_{\overline{m}} &= \frac{n-1}{n+1} \\ p &= 2\frac{\overline{v}_m}{n+1} \end{aligned} \tag{7}$$

2. If $\overline{v}_m \in [\underline{\rho}(n)\overline{v}_M, \overline{\rho}(n)\overline{v}_M]$*,* $q_{\overline{M}}$*,* $q_{\overline{m}}$*, and* p *satisfy:*

$$\begin{aligned} q_{\overline{m}} + q_{\overline{M}} &= \frac{2n}{n+1} \\ p &= \frac{2q_{\overline{m}}\overline{v}_M}{2(n-1)-(n-3)q_{\overline{m}}} \\ p &= \frac{2(2-q_{\overline{M}})\overline{v}_m}{2(n-1)-(n-3)q_{\overline{M}}}. \end{aligned} \tag{8}$$

3. If $\overline{v}_m \in [\overline{\rho}(n)\overline{v}_M, \overline{v}_M]$*,* $q_{\overline{M}}$*,* $q_{\overline{m}}$*, and* p *satisfy:*

$$\begin{aligned} q_{\overline{m}} &= 1 \\ q_{\overline{M}} &= \frac{n-1}{(n+1)} \\ p &= 2\frac{\overline{v}_M}{n+1} \end{aligned} \tag{9}$$

The two thresholds $\underline{\rho}(n)$ *and* $\overline{\rho}(n)$ *are given by:*

$$\begin{aligned} \underline{\rho}(n) &= \frac{n+1}{n+5} \\ \overline{\rho}(n) &= \frac{(n-1)(n+5)}{(n+3)(n+1)} \end{aligned} \tag{10}$$

(b) Case $n = 3$

1. If $v_{(2)M} \leq \frac{3}{4}\overline{v}_M$*, then* $\mu(3)v_{(2)M} \leq \underline{\rho}(3)\overline{v}_M$*, and the characterization in part (a) above applies unchanged. If* $v_{(2)M} > \frac{3}{4}\overline{v}_M$*, then:*

2. If $\overline{v}_m \in [\mu(3)v_{(2)M}, \overline{\rho}(3)\overline{v}_M]$*,* $q_{\overline{M}}$*,* $q_{\overline{m}}$*, and* p *satisfy system 8; if* $\overline{v}_m \in [\overline{\rho}(3)\overline{v}_M, \overline{v}_M]$*,* $q_{\overline{M}}$*,* $q_{\overline{m}}$*, andp satisfy system 9.*

Lemma 3. *Suppose* $G = m$ *(or* $\overline{v}_G = \overline{v}_m$ *and* $\overline{v}_g = \overline{v}_M$*). Then if* $\overline{v}_M \in \left[\mu(n)v_{(2)m}, \overline{v}_m\right]$*, where* $\mu(n)$ *is given by relation 6 above, the strategies described in the theorem, together with the price and mixing probabilities given by system 9 are a fully revealing ex ante competitive equilibrium for all* n *odd,* m*, and* F*.*

The proof is organized in two stages. First, we show that if the direction of preferences associated with each demand is commonly known, the strategies and price described above are an equilibrium. Second, we show that when preferences are private information the equilibrium is fully revealing.

Ex ante equilibrium with full information

Suppose first that preferences are publicly known. We show here that the three systems 7, 8, and 9 characterize an ex ante equilibrium for each corresponding range of realized valuations.

1. Consider a candidate equilibrium with $q_{\overline{M}} \in (0,1)$, $q_{\overline{m}} \in (0,1)$. Expected market balance requires $(q_{\overline{M}} + q_{\overline{m}})(n-1)/2 = (n-2) + (1-q_{\overline{M}}) + (1-q_{\overline{m}})$, or:

$$q_{\overline{M}} + q_{\overline{m}} = \frac{2n}{n+1} \tag{11}$$

Denote by $U_M(s)$ the expected utility to voter $\overline{v}_M$ from demand s. Then:

$$\begin{aligned} U_M\left(\frac{n-1}{2}\right) &= q_{\overline{m}}\left(\frac{\overline{v}_M}{2} - \frac{n-1}{4}p\right) + (1-q_{\overline{m}})\left(\overline{v}_M - \frac{n-1}{2}p\right) \\ U_M(-1) &= q_{\overline{m}}\left(\frac{p}{2}\right) + (1-q_{\overline{m}})(\overline{v}_M) \end{aligned}$$

where we are assuming that voter $\overline{v}_M$ is informed that the other voter randomizing with probability $q_{\overline{m}}$ belongs to the minority. Voter $\overline{v}_M$ is indifferent between the two pure demands if and only if:

$$p = \frac{2q_{\overline{m}}\overline{v}_M}{n+1+(n-3)(1-q_{\overline{m}})} \tag{12}$$

Similarly, the indifference condition for voter $\overline{v}_m$ requires

$$p = \frac{2(2-q_{\overline{M}})\overline{v}_m}{n+1+(n-3)q_{\overline{M}}} \tag{13}$$

Equations 11, 12 and 13 corresponds to system 7 in Lemma 2. The existence of a solution is not guaranteed. There is a solution if and only if there exists $q_{\overline{M}} \in [0,1]$ and $q_{\overline{m}} \in [0,1]$ with $q_{\overline{M}} + q_{\overline{m}} = \frac{2n}{n+1}$ such that (12)=)13). Such conditions are satisfied if and only if $\overline{v}_m \in [\underline{\rho}(n)\overline{v}_M, \bar{\rho}(n)\overline{v}_M]$ where $\underline{\rho}(n) = \frac{n+1}{n+5}$ and $\bar{\rho}(n) = \frac{(n-1)(n+5)}{(n+3)(n+1)}$.

Note that $\frac{1}{2} \leq \underline{\rho}(n) < \bar{\rho}(n) < 1$ for all $n \geq 3$. To verify that this is indeed an equilibrium, we need to rule out profitable deviations.First, for any voter, any demand

$s_i > n-1$ is always fully rationed, and thus is equivalent to $s_i = 0$.

(i) Consider first voter $\overline{v}_M$. For any $s_M \in (\frac{n-1}{2}, n-1]$, $U_M(s_M) < U_M(\frac{n-1}{2})$: demanding more votes than required to achieve a strict majority does not affect the probability of rationing and is strictly costly. For any $s_M \in [0, \frac{n-1}{2})$, $U_M(s_M) < U_M(-1)$: demanding less than $\frac{n-1}{2}$ votes is dominated by selling. To see this, note that when $s_m = \frac{n-1}{2}$, any $s_M < \frac{n-1}{2}$ guarantees that $\overline{v}_m$ will not be rationed and will win (because all other voters are selling). Thus, whether $s_M \in (0, \frac{n-1}{2})$ and the action is strictly costly, or $s_M = 0$ and voter $\overline{v}_M$ stays out of the market, when $s_m = \frac{n-1}{2}$, any $s_M \in [0, \frac{n-1}{2})$ is strictly dominated by selling. When $s_m = -1$, any $s_M \in (0, \frac{n-1}{2}]$ is dominated by $s_M \in \{-1, 0\}$ and these two actions are equivalent because both $s_M = -1$ and $s_M = 0$ induce no trade and guarantee a majority victory. Therefore, when facing the strategy profile defined in the candidate equilibrium, $\overline{v}_M$'s best response can only be either $s_M = -1$ or $s_M = \frac{n-1}{2}$. System 7 guarantees that $\overline{v}_M$ is indifferent between the two demands.

(ii) Consider now voter $\overline{v}_m$. As above, for any $s_m \in \left(\frac{n-1}{2}, n-1\right]$, $U_m(s_m) < U_m(\frac{n-1}{2})$. It is also clear that $U_m(0) < U_m(-1)$: the two demands are equivalent if $s_M = -1$ and selling is strictly superior to staying out of the market if $s_M = \frac{n-1}{2}$. The question is whether $\overline{v}_m$ could gain by demanding less than $\frac{n-1}{2}$ votes. Consider the relevant expected utilities:

$$\begin{aligned} U_m\left(\frac{n-1}{2}\right) &= (1-q_{\bar{M}})\left(\bar{v}_m - \frac{n-1}{2}p\right) + q_{\bar{M}}\left(\frac{\bar{v}_m}{2} - \frac{n-1}{4}p\right) \\ U_m(-1) &= (1-q_{\bar{M}})\cdot 0 + q_{\bar{M}}\left(\frac{p}{2}\right) \\ U_m(x) &= (1-q_{\bar{M}})\left(P(x)\bar{v}_m - xp\right) + q_{\bar{M}}\left(-xp\right) \end{aligned} \tag{14}$$

where $P(x)$ is the probability of a minority victory when $\overline{v}_m$ demands $x \in (0, \frac{n-1}{2})$ votes and $\overline{v}_M$ offers his vote for sale. Since $P(x) < 1$ for all $x \in (0, \frac{n-1}{2})$, and $U_m(x)$ is increasing in $P(x)$ and decreasing in x, it follows that $U_m(x) < (1 - q_{\bar{M}})(\overline{v}_m - p) + q_{\bar{M}}(-p)$. Hence $U_m\left(\frac{n-1}{2}\right) > (1-q_{\bar{M}})(\overline{v}_m - p) + q_{\bar{M}}(-p)$ is sufficient to rule out a profitable deviation to $x \in (0, \frac{n-1}{2})$. The condition is equivalent to:

$$\frac{q_{\bar{M}}}{2}\bar{v}_m \geq \frac{2(1-q_{\bar{M}})(n-1) + q_{\bar{M}}(n-1) - 4}{4}p$$

Substituting p from (13) and simplifying, the condition amounts to:

$$(2-n)q_{\bar{M}}^2 + (3n-5)q_{\bar{M}} - 2n + 6 \geq 0$$

This function is increasing in $q_{\bar{M}}$ for all $n \geq 3$. By equation 11, $q_{\bar{M}} \geq \frac{n-1}{n+1}$. Hence, we can evaluate the condition at $q_{\bar{M}} = \frac{n-1}{n+1}$. If it is positive, the deviation is not profitable. Substituting, we obtain $n^2 + 2n + 13 \geq 0$, which is trivially satisfied for all n. Hence for any $s_m \in [1, \frac{n-1}{2})$, $U_m(s_m) < U_m(\frac{n-1}{2})$. We can conclude that when facing the strategy profile defined in the candidate equilibrium, $\overline{v}_m$'s best response can only be either $s_M = -1$ or $s_M = \frac{n-1}{2}$. System 7 guarantees that $\overline{v}_m$ is indifferent between them.

(iii) Consider $v_i \in M$, $v_i \neq \overline{v}_M$. We show here that, given others' specified strategies, v_i's best response is selling: $s_i = -1$. First notice that, as argued above and for the same reasons, $U_i(s_i) < U_i(\frac{n-1}{2})$ for any $s_i \in (\frac{n-1}{2}, n-1]$. We need to treat the cases $n \geq 5$ and $n = 3$ separately.

(iii.a) Suppose first $n > 3$. In this case, for the same reasons described above $U_i(0) < U_i(-1)$. If a deviation from $s_i = -1$ is profitable, it must be to some $s_i \in (0, \frac{n-1}{2}]$. Suppose first $s_M = -1$. Then in the candidate equilibrium the profile of others' strategies faced by v_i is identical to the profile faced by $\overline{v}_M$. In particular, $U_i(-1) = U_M(-1) = U_M(\frac{n-1}{2}) > U_M(s)$ for all $s \in [0, \frac{n-1}{2})$. But $U_i(s)$ is increasing in v_i for all $s \in (0, \frac{n-1}{2}]$; hence for all s in this interval $U_i(s) < U_M(s)$, and thus $U_i(-1) > U_i(s)$ for all $s \in (0, \frac{n-1}{2}]$. Thus if $s_M = -1$, $s_i = -1$ is v_i's best response. Suppose then $s_M = \frac{n-1}{2}$. For all $s_i \in [0, \frac{n-3}{2})$, v_i is never rationed, but there is always another voter, either $\overline{v}_M$ or $\overline{v}_m$, who exits the market holding a majority of the votes. Hence the strategy is costly for v_i and never increases the probability of his side winning. It is dominated by $s_i = -1$. Consider then the two remaining strategies $s_i = \frac{n-1}{2}$, and $s_i = \frac{n-3}{2}$. Conditional on $s_M = \frac{n-1}{2}$, the relevant expected utilities are:

$$\begin{aligned}
U_{i\in M}\left(\frac{n-1}{2}\right)\Bigg|_{s_M=\frac{n-1}{2}} &= (1-q_{\overline{m}})\left(v_i - \frac{n-1}{4}p\right) + q_{\overline{m}}\left(\frac{2v_i}{3} - \frac{n-1}{6}p\right) \\
U_{i\in M}\left(\frac{n-3}{2}\right)\Bigg|_{s_M=\frac{n-1}{2}} &= (1-q_{\overline{m}})\left(v_i - \frac{n-3}{2}p\right) + q_{\overline{m}}\left(\frac{2v_i}{3} - \frac{n-3}{6}p\right) \\
U_{i\in M}(-1)\Bigg|_{s_M=\frac{n-1}{2}} &= (1-q_{\overline{m}})\left(v_i + \frac{p}{2}\right) + q_{\overline{m}}\left(\frac{v_i}{2} + \frac{n-1}{2(n-2)}p\right)
\end{aligned} \tag{15}$$

Taking into account $q_{\overline{m}} \in \left[\frac{n-1}{n+1}, 1\right]$, equation 12, and $v_i \leq \overline{v}_M$, it is then straightforward to show that, conditional on $s_M = \frac{n-1}{2}$, $U_{i\in M}(-1) > U_{i\in M}\left(\frac{n-1}{2}\right)$, and $U_{i\in M}(-1) > U_{i\in M}\left(\frac{n-3}{2}\right)$. But if $s_i = -1$ is v_i's best response both when $s_M = -1$ and when $s_M = \frac{n-1}{2}$, than it is v_i's best response when $\overline{v}_M$ randomizes between $s_M = -1$ and $s_M = \frac{n-1}{2}$. No profitable deviation exists.

(iii.b) Suppose now $n = 3$. There are two M voters; hence $v_i \in M$, $v_i \leq \overline{v}_M$, is $v_{(2)M}$, the M voter with second highest value. This case must be considered separately because if $n = 3$, and only if $n = 3$, $v_{(2)M}$ can induce no trade with probability $q_{\overline{m}}q_{\overline{M}}$ by unilaterally deviating and staying out of the market. Conditional on $s_M = \frac{n-1}{2} = 1$, the relevant expected utilities are:

$$\begin{aligned} U_{(2)M}(1)\Big|_{s_M=1} &= (1-q_{\overline{m}})\left(v_i - \frac{n-1}{4}p\right) + q_{\overline{m}}v_i \\ U_{(2)M}(0)\Big|_{s_M=1} &= v_i \\ U_{(2)M}(-1)\Big|_{s_M=1} &= (1-q_{\overline{m}})\left(v_i + \frac{p}{2}\right) + q_{\overline{m}}\left(\frac{v_i}{2} + p\right) \end{aligned} \tag{16}$$

It is immediately clear that $U_{(2)M}(0) > U_{(2)M}(1)$. Given equations 13 and 11, $U_{(2)M}(-1) > U_{(2)M}(0)$ for all $\overline{v}_m \in [\underline{\rho}(3)\overline{v}_M, \bar{\rho}(3)\overline{v}_M] \iff \overline{v}_m > (2/3)v_{(2)M}$. Thus $s_i = -1$ is indeed a best response for $v_{(2)M}$ as long as

$$\overline{v}_m \in \left[\max\{(2/3)v_{(2)M}, \underline{\rho}(3)\overline{v}_M\}, \bar{\rho}(3)\overline{v}_M\right]$$

.

(iv) Finally, consider $v_i \in m$, $v_i \neq \overline{v}_m$. Note that such a voter only exists for $n > 3$. Again, we show here that, given others' specified strategies, v_i's best response is selling: $s_i = -1$. The proof proceeds as above. First notice that, as above, $U_i(s_i) < U_i(\frac{n-1}{2})$ for any $s_i \in (\frac{n-1}{2}, n-1]$, and $U_i(0) < U_i(-1)$. If a deviation from $s_i = -1$ is profitable, it must be to some $s_i \in (0, \frac{n-1}{2}]$. Suppose first $s_m = -1$. Then in the candidate equilibrium the profile of others' strategies faced by v_i is identical to the profile faced by $\overline{v}_m$. In particular, $U_i(-1) = U_m(-1) = U_m(\frac{n-1}{2}) > U_m(s)$ for all $s \in [0, \frac{n-1}{2})$. But $U_i(s)$ is increasing in v_i for all $s \in (0, \frac{n-1}{2}]$; hence for all s in this interval $U_i(s) < U_m(s)$, and thus $U_i(-1) > U_i(s)$ for all $s \in (0, \frac{n-1}{2}]$. Thus if $s_m = -1$, $s_i = -1$ is v_i's best response. Suppose then $s_m = \frac{n-1}{2}$. Exactly as argued above, if $s_i \in [0, \frac{n-3}{2})$, v_i is never rationed, but there is always another voter, either $\overline{v}_M$ or $\overline{v}_m$, who exits the market holding a majority of the votes. Hence the strategy is costly for v_i and never increases the probability of his side winning. It is dominated by $s_i = -1$. Consider then the two remaining strategies $s_i = \frac{n-1}{2}$, and $s_i = \frac{n-3}{2}$. Conditional on $s_m = \frac{n-1}{2}$, the relevant expected utilities are:

$$\begin{aligned}
U_{i\in m}\left(\frac{n-1}{2}\right)\Big|_{s_m = \frac{n-1}{2}} &= (1-q_{\bar{M}})\left(v_i - \frac{n-1}{4}p\right) + q_{\bar{M}}\left(\frac{2v_i}{3} - \frac{n-1}{6}p\right) \\
U_{i\in m}\left(\frac{n-3}{2}\right)\Big|_{s_m = \frac{n-1}{2}} &= (1-q_{\bar{M}})\left(v_i - \frac{n-3}{2}p\right) + q_{\bar{M}}\left(\frac{2v_i}{3} - \frac{n-3}{6}p\right) \\
U_{i\in m}(-1)\Big|_{s_m = \frac{n-1}{2}} &= (1-q_{\bar{M}})\left(v_i + \frac{p}{2}\right) + q_{\bar{M}}\left(\frac{v_i}{2} + \frac{n-1}{2(n-2)}p\right)
\end{aligned}$$

Taking into account $q_{\overline{M}} \in \left[\frac{n-1}{n+1}, 1\right]$, equation 13, and $v_i \leq \overline{v}_m$, it is then straightforward to show that, conditional on $s_m = \frac{n-1}{2}$, $U_{i\in m}(-1) > U_{i\in m}\left(\frac{n-1}{2}\right)$, and $U_{i\in m}(-1) > U_{i\in m}\left(\frac{n-3}{2}\right)$. But if $s_i = -1$ is v_i's best response both when $s_m = -1$ and when $s_m = \frac{n-1}{2}$, than it is v_i's best response when $\overline{v}_m$ randomizes between $s_m = -1$ and $s_m = \frac{n-1}{2}$. No profitable deviation exists. We can conclude that if $\overline{v}_m \in [\max\{\mu(n)v_{(2)M}, \underline{\rho}(n)\overline{v}_M\}, \bar{\rho}(n)\overline{v}_M]$, where $\mu(n)$ is given by equation 6, and $\underline{\rho}(n)$ and $\bar{\rho}(n)$ are given by system 10, the strategies described in the theorem, together with the price and the mixing probabilities characterized in system 8, are indeed an ex ante equilibrium of the full information game. Note that $\underline{\rho}(n)\overline{v}_M > \mu(n)v_{(2)M}$ for all $n > 3$; if $n = 3$, $\underline{\rho}(3)\overline{v}_M > (2/3)v_{(2)M} \iff v_{(2)M} < (3/4)\overline{v}_M$.

2. Consider now $\overline{v}_m \in [\mu(n)v_{(2)M}, \underline{\rho}(n)\overline{v}_M]$, where $\mu(n)$ is given by relation 6. Note that this case is relevant if $\underline{\rho}(n)\overline{v}_M > \mu(n)v_{(2)M}$, and thus for all $n > 3$, or for $v_{(2)M} < (3/4)\overline{v}_M$ if $n = 3$. Suppose all voters adopt the strategies described in the theorem, and $q_{\overline{M}} = 1$. Expected market clearing (equation 11) implies $q_{\overline{m}} = \frac{n-1}{n+1}$, and $U_m(-1) = U_m(\frac{n-1}{2})$ (or equation 13) implies $p = \frac{2\overline{v}_m}{n+1}$. Thus suppose system 8 holds. We show here that such strategies and price are an ex ante equilibrium of the full information game. As above, we rule out any profitable deviation for each voter in turn. Again, note that for any voter any demand $s_i > n-1$ is always fully rationed, and thus is equivalent to $s_i = 0$.

(i) Consider first voter $\overline{v}_M$. In the candidate equilibrium, $s_M = \frac{n-1}{2}$. As argued earlier, it remains true that for any $s_M \in (\frac{n-1}{2}, n-1]$, $U_M(s_M) < U_M(\frac{n-1}{2})$: demanding more votes than required to achieve a strict majority does not affect the probability of rationing and is strictly costly. Similarly, it remains true that for any $s_M \in [0, \frac{n-1}{2})$, $U_M(s_M) < U_M(-1)$: demanding less than $\frac{n-1}{2}$ votes is dominated by selling. The argument is identical to what described earlier. Thus the only deviation we need to consider is to $s_M = -1$. The relevant expected utilities are:

$$\begin{aligned} U_M\left(\frac{n-1}{2}\right) &= q_{\bar{m}}\left(\frac{\overline{v}_M}{2} - \frac{n-1}{4}p\right) + (1-q_{\bar{m}})\left(\overline{v}_M - \frac{n-1}{2}\right) \\ U_M(-1) &= q_{\bar{m}}\left(\frac{p}{2}\right) + (1-q_{\bar{m}})(\overline{v}_M) \end{aligned}$$

Substituting $q_{\overline{m}} = \frac{n-1}{n+1}$ and $p = \frac{2\overline{v}_m}{n+1}$, we obtain:

$$U_M\left(\frac{n-1}{2}\right) \geq U_M(-1) \Leftrightarrow \overline{v}_M \geq \frac{n+5}{n+1}\overline{v}_m = \frac{1}{\underline{\rho}(n)}\overline{v}_m$$

The requirement established the upper bound of the range of $\overline{v}_m$ values considered here: $\overline{v}_m \in [\mu(n)v_{(2)M}, \underline{\rho}(n)\overline{v}_M]$.

(ii) Consider voter $\overline{v}_m$. The arguments discussed under point 1.(ii) apply. With $s_M = \frac{n-1}{2}$ and all other voters selling, $s_m = \frac{n-1}{2}$ and $s_m = -1$ dominate all other $v_{\overline{m}}$'s strategies. With $p = \frac{2\overline{v}_m}{n+1}$, $\overline{v}_m$ is indifferent between them and has no profitable deviation.

(iii) Consider now $v_i \in M$, $v_i \neq \overline{v}_M$. We show here that, given others' specified strategies, v_i's best response is selling: $s_i = -1$. By the arguments under point 1.(iii) above, the only deviations we need to consider are $s_i = \frac{n-1}{2}$ and $s_i = \frac{n-3}{2}$. The relevant expected utilities are given by system 15 for $n > 3$, and system 16 for $n = 3$. Substituting $p = \frac{2\overline{v}_m}{n+1}$, and $q_{\overline{m}} = \frac{n-1}{n+1}$, we derive the following conditions. If $n > 3$:

$$U_{i\in M}\left(\frac{n-1}{2}\right) \leq U_{i\in M}(-1) \Leftrightarrow v_i\frac{(n-2)(n-1)}{2(n^2+n-5)} \leq \overline{v}_m$$

and

$$U_{i\in M}\left(\frac{n-3}{2}\right) \leq U_{i\in M}(-1) \Leftrightarrow v_i\frac{(n-2)(n-1)(n+1)}{2(n^3+3n^2-19n+21)} \leq \overline{v}_m$$

The two conditions are satisfied if and only if $\mu(n)v_i \leq \overline{v}_m$. Thus they are satisfied for all $v_i \in M$, $v_i \leq \overline{v}_M$ if they are satisfied for $v_i = v_{(2)M}$. If $n = 3$:

$$U_{(2)M}(1) \leq U_{(2)M}(-1) \Leftrightarrow \frac{v_{(2)M}}{2} \leq \overline{v}_m$$

and:

$$U_{(2)M}(0) \leq U_{(2)M}(-1) \Leftrightarrow \frac{2}{3}v_{(2)M} \leq \overline{v}_m$$

This latter condition is stricter and again is satisfied if and only if $\mu(3)v_{(2)M} \leq \overline{v}_m$. For all n, we have established the lower bound of the range of $\overline{v}_m$ values considered here: $\overline{v}_m \in [\mu(n)v_{(2)M}, \underline{\rho}(n)\overline{v}_m]$. Recall that $\underline{\rho}(n)\overline{v}_M > \mu(n)v_{(2)M}$ for all $n > 3$, but if $n = 3$, $\underline{\rho}(3)\overline{v}_M > \mu(3)v_{(2)M} \iff v_{(2)M} < (3/4)\overline{v}_M$ if $n = 3$.

(iv) Finally, consider $v_i \in m$, $v_i \neq \overline{v}_m$. Again, this voter only exists if $n > 3$. The arguments in 1.(iv) above can be applied identically here and establish that $s_i = -1$ is v_i's unique best response. In particular, if $s_m = -1$, the profile of others' strategies faced by v_i is identical to the profile faced by $\overline{v}_m$. Given others' specified strategies, the differential utility from selling, relative to any other action, is decreasing in v_i; hence if $s_m = -1$ is $\overline{v}_m$'s best response, then it must be a best response for $v_i \leq \overline{v}_m$. If $s_m = \frac{n-1}{2}$, the identical proof detailed in 1.(iv) is relevant. The proof made use of the constraint $q_{\bar{M}} \in \left[\frac{n-1}{n+1}, 1\right]$, which is still satisfied here.

We conclude that for all $\overline{v}_m \in \left[\mu(n)v_{(2)M}, \underline{\rho}(n)\overline{v}_M\right]$, where $\mu(n)$ is given by relation 6, the strategies described in the theorem, together with the price and the mixing probabilities characterized in system 7, are indeed an ex ante equilibrium of the full information game. If $n = 3$, this case is only relevant if $v_{(2)M} < \frac{3}{4}\overline{v}_M$.

3. Consider now $\overline{v}_m > \bar{\rho}(n)\overline{v}_M$, where $\bar{\rho}(n)$ is defined in system 10. Suppose all voters adopt the strategies described in the theorem, and $q_{\overline{m}} = 1$. Expected market clearing (equation 11) implies $q_{\overline{M}} = \frac{n-1}{n+1}$, and $U_M(-1) = U_M(\frac{n-1}{2})$ (or equation 12) implies $p = \frac{2\overline{v}_M}{n+1}$. Thus suppose system 9 holds. We show here that such strategies and price are an ex ante equilibrium of the full information game. As above, we rule out any profitable deviation for each voter in turn. The proofs follow immediately from the arguments used earlier. In particular:

(i) Consider first voter $\overline{v}_M$. The arguments discussed under point 1.(i) apply. With

$s_m = \frac{n-1}{2}$ and all other voters selling, $s_M = \frac{n-1}{2}$ and $s_M = -1$ dominate all other $\overline{v}_M$'s strategies. With $p = \frac{2\overline{v}_M}{n+1}$, $\overline{v}_M$ is indifferent between them and has no profitable deviation.

(ii) Consider then voter $\overline{v}_m$. Recall that when $\overline{v}_M$ randomizes between $s_M = \frac{n-1}{2}$ and $s_M = -1$ and all others sell, $s_m = \frac{n-1}{2}$ and $s_m = -1$ dominate all other $\overline{v}_m$'s strategies. Substituting $q_{\overline{M}} = \frac{n-1}{n+1}$ and $p = \frac{2\overline{v}_M}{n+1}$ in the expected utility of the minority leader for the two possibilities yield:

$$U_m\left(\frac{n-1}{2}\right) \geq U_m(-1) \Leftrightarrow \overline{v}_m \geq \frac{(n-1)(n+5)}{(n+1)(n+3)}\overline{v}_M = \bar{\rho}(n)\overline{v}_M$$

The condition establishes the lower bound of the range of $\overline{v}_m$ values considered under this case.

(iii) Consider $v_i \in M$, $v_i \neq \overline{v}_M$. If $n > 3$, the arguments in 1.(iii.a) above can be applied identically here and establish that $s_i = -1$ is v_i's unique best response. In particular, if $s_M = -1$, the profile of others' strategies faced by v_i is identical to the profile faced by $\overline{v}_M$. Hence if $s_M = -1$ is $\overline{v}_M$'s best response, then it must be a best response for $v_i \leq \overline{v}_M$. If $s_M = \frac{n-1}{2}$, the identical proof detailed in 1.(iii) is relevant. The proof made use of the constraint $q_m \in \left[\frac{n-1}{n+1}, 1\right]$, which is still satisfied here. If $n = 3$, $v_i \equiv v_{(2)M}$ and:

$$\begin{aligned} U_{(2)M}(1)\Big|_{s_m=1} &= q_{\overline{M}} v_{(2)M} + (1-q_{\overline{M}})\left(\frac{v_{(2)M}}{2} + \frac{p}{2}\right) \\ U_{(2)M}(0)\Big|_{s_m=1} &= q_{\overline{M}} v_{(2)M} \\ U_{(2)M}(-1)\Big|_{s_m=1} &= q_{\overline{M}}\left(\frac{v_{(2)M}}{2} + p\right) + (1-q_{\overline{M}})\left(\frac{p}{2}\right) \end{aligned}$$

With $p = \frac{2\overline{v}_M}{n+1}$ and $q_{\overline{M}} = \frac{1}{2}$ by equation 11, it is trivial to verify that $U_{(2)M}(-1) > U_{(2)M}(1)$ and $U_{(2)M}(-1) > U_{(2)M}(0)$.

(iv) Finally, when $n > 3$, consider $v_i \in m$, $v_i \neq \overline{v}_m$. The problem faced here by $v_i \in m$ is identical to the problem faced by $v_i \in M$, $v_i \neq \overline{v}_M$ in case 2.(iii) above, when $q_{\overline{M}} = 1$, $q_{\overline{m}} = \frac{n-1}{n+1}$. Taking into account $p = \frac{2\overline{v}_M}{n+1}$, all profitable deviations can be ruled out if and only if $v_i \max\left\{\frac{(n-2)(n-1)}{2(n^2+n-5)}, \frac{(n-2)(n-1)(n+1)}{2(n^3+3n^2-19n+21)}\right\} \leq \overline{v}_M$, or $v_i\mu(n) \leq \overline{v}_M$.

Because $\mu(n) < 1$, two observations follow immediately. First, if $\overline{v}_M \geq \overline{v}_m$, the condition $v_i\mu(n) \leq \overline{v}_M$ for all $v_i \in m$, $v_i \neq \overline{v}_m$ is always satisfied. Thus the strategies described in the theorem, together with the price and mixing probabilities characterized in system 9 are indeed an ex ante equilibrium of the full information game for all $\overline{v}_m \in (\bar{\rho}(n)\overline{v}_M, \overline{v}_M]$. Second, the condition $\overline{v}_M \geq \overline{v}_m$ has not been imposed anywhere in the proof of the equilibrium of case 3. The equilibrium requires $\overline{v}_m > \bar{\rho}(n)\overline{v}_M$, where $\bar{\rho}(n) < 1$, and, for $n > 3$, $v_i\mu(n) \leq \overline{v}_M$ $\forall v_i \in m$, $v_i \neq \overline{v}_m$. Thus it is compatible with $\overline{v}_m > \overline{v}_M$, as long as $\overline{v}_M \geq \mu(n)v_{(2)m}$ if $n > 5$, and with no additional constraint if $n = 3$. Hence Lemma 3 follows immediately.

We now show that when preferences are private information, the strategies and price identified above constitute a fully revealing ex ante equilibrium.

Fully revealing equilibrium

We conjecture an equilibrium identical to the full information equilibrium characterized above and show that given others' strategies, the equilibrium price and the knowledge that the market is in a fully revealing equilibrium, each voter's best response when preferences are private information is uniquely identified and equals the voter's best response with full information. Thus the equilibrium exists when preferences are private information and is indeed fully revealing.

1. Consider first the perspective of voter $\overline{v}_M$, in equilibrium. In any of the scenarios identified above, expected market equilibrium requires $\overline{v}_M$ to demand a positive number of votes with positive probability. It then follows that the other voter who demands a positive number of votes with positive probability must belong to the minority. If not, $\overline{v}_M$'s best response would be to sell, violating expected market equilibrium. Thus $\overline{v}_M$ also knows that $M - 1$ majority members and $m - 1$ minority members are offering their vote for sale; he cannot identify them individually, but that is irrelevant. Given that the other net demand for votes comes from a minority voter, $\overline{v}_M$'s best response is identified uniquely and is identical to his best response under full information.

2. Consider then the perspective of voter $\overline{v}_m$. If $n = 3$, he is the only minority voter and the problem is trivial. Suppose $n > 3$. Suppose first that $\overline{v}_m \in \left[\mu(n)v_{(2)M}, \underline{\rho_M}(n)\overline{v}_M\right]$, and hence $s_M = \frac{n-1}{2}$ with probability 1. Expected market balance requires $\overline{v}_m$ to demand a positive number of votes with positive probability. But that can only be a best response if the voter who demands $\frac{n-1}{2}$ votes belongs to the majority; if not, $\overline{v}_m$'s best response would be to sell. Again, $\overline{v}_m$ also knows that $M - 1$ majority members and $m - 1$ minority members are offering their vote for sale; he cannot identify them

individually, but that is irrelevant. Suppose now $\overline{v}_m \in [\underline{\rho}(n)\overline{v}_M, \bar{\rho}(n)\overline{v}_M]$. Expected market balance rules out that $\overline{v}_m$ could sell with probability 1 (because over this range of valuations the minimal expected demand of votes by $\overline{v}_m$ required for expected market balance is $\min(q_{\overline{m}})(\frac{n-1}{2})+(1-\min(q_{\overline{m}})(-1) = \left(\frac{n-1}{n+1}\right)\left(\frac{n-1}{2}\right)+\left(1-\frac{n-1}{n+1}\right)(-1) = \frac{n-5}{2(n+1)} >$ -1 for all $n \geq 3$). Given the profile of strategies faced by $\overline{v}_m$, staying out of the market ($s_m = 0$) is always dominated by selling. Thus $\overline{v}_m$'s best response in equilibrium must include demanding a positive number of votes with positive probability. As in all previous cases, demanding more than $\frac{n-1}{2}$ votes is always dominated by demanding $\frac{n-1}{2}$ votes. Thus the actions over which $\overline{v}_m$ can randomize with positive probability are $s_m = \frac{n-1}{2}$, $s_m = x$, with $0 \leq x < \frac{n-1}{2}$, and $s_m = -1$. Suppose that the voter demanding $\frac{n-1}{2}$ with probability $q_{\overline{M}}$ (with $q_{\overline{M}}$ identified in system 7), and selling otherwise, belonged to the minority. Then:

$$
\begin{aligned}
U_m\left(\frac{n-1}{2}\right)\Big|_{(\overline{v}_M \in m)_e} &= (1-q_{\bar{M}})\left(\overline{v}_m - \frac{n-1}{2}p\right) + q_{\bar{M}}\left(\overline{v}_m - \frac{n-1}{4}p\right) \\
U_m(-1)\Big|_{(\overline{v}_M \in m)_e} &= (1-q_{\bar{M}})\cdot 0 + q_{\bar{M}}\left(\overline{v}_m + \frac{p}{2}\right) \\
U_m(x)\Big|_{(\overline{v}_M \in m)_e} &= (1-q_{\bar{M}})\left(P(x)\overline{v}_m - xp\right) + q_{\bar{M}}\left(\overline{v}_m - xp\right)
\end{aligned}
\tag{17}
$$

where the index $(\overline{v}_m \in m)_e$ indicates the belief that the other voter with positive expected demand belongs to the minority. System 17 is similar to system 14. In particular: (1) The differential utility from selling relative to demanding $x \in [0, \frac{n-1}{2})$ votes, $U_m(-1) - U_m(x)$, is identical. We saw earlier that such term must be positive for all $q_{\bar{M}} \in \left[\frac{n-1}{n+1}, 1\right]$, a result that thus applies immediately here. (2) For all $\overline{v}_m > 0$, the differential utility from selling relative to demanding $\frac{n-1}{2}$ votes, $U_m(-1) - U_m\left(\frac{n-1}{2}\right)$, is strictly higher than in system 14, where, at equilibrium $q_{\bar{M}}$, it equalled 0. Hence at equilibrium $q_{\bar{M}}$ it must be positive here. It follows that if the voter demanding $\frac{n-1}{2}$ with probability $q_{\overline{M}}$ belonged to the minority, $\overline{v}_m$'s best response would be to sell. But that would violate expected market balance. Hence the voter demanding $\frac{n-1}{2}$ with probability $q_{\overline{M}}$ must belong to the majority. Of all remaining voters offering their votes for sale, $M-1$ belongs to the majority, and $m-1$ to the minority. They cannot be distinguished but that has no impact on $\overline{v}_m$'s unique best response. Finally, suppose either $\overline{v}_m \in (\bar{\rho}(n)\overline{v}_M, \overline{v}_M]$, or $\overline{v}_M \in \left[\mu(n)v_{(2)m}, \overline{v}_m\right]$. Expected market balance requires $s_m = \frac{n-1}{2}$ with probability 1. But then the other voter demanding $\frac{n-1}{2}$ votes with positive probability cannot belong to the minority (because in a fully revealing equilibrium, if $s_m = \frac{n-1}{2}$ with probability 1, all other minority voters would prefer to sell). Hence again the other voter with positive demand for votes must be a majority voter. All remaining voters are sellers; identifying the group each of them belongs to is not possible but has no impact on $\overline{v}_m$'s unique best response.

3. Consider now the perspective of all voters who in the full information equilibrium offer their vote for sale with probability 1: $v_i \in M$, $v_i \neq \overline{v}_M$, or $v_i \in m$, $v_i \neq \overline{v}_m$. By the arguments above, each of them knows that in a fully revealing equilibrium the two voters with positive expected demand must belong to the two different parties. Which one belongs to the majority and which one to the minority cannot be distinguished, but is irrelevant: since in the full information case v_i's best response is $s_i = -1$ with probability 1 whether $v_i \in M$, or $v_i \in m$, it follows that identifying which of the two voters with positive expected demand belongs to which group is irrelevant to v_i's best response. Equally irrelevant is identifying which of the sellers belongs to which group. Although the direction of preferences associated with each individual voter cannot be identified, v_i's best response is unique and identical to his best response with full information.

We can conclude that the equilibrium strategies and price identified by Lemmas 2 and 3 are indeed a fully revealing ex ante equilibrium with private information. ∎

A.2 Proof of the Corollary to Theorem 2

Corollary 1. *For any* $\alpha \in \left(0, \frac{1}{2}\right)$ *and* F, $\Pr[\lim_{n\to\infty} q_{\overline{G},n}(\mathbf{v}) = 1] = 1$, *and* $\Pr[\lim_{n\to\infty} q_{\overline{G},n}(v) = 1] = 1$

Proof. For $h = g, G$, define $q_{\overline{h},n}(\mathbf{v})$ as a sequence of random variables that take the values specified in Theorem 1 if the condition in the theorem is satisfied, and 0 otherwise. We will use the Borel Cantelli lemma. In the context of almost sure convergence, it implies that a sufficient condition for a sequence of random variable X_n to converge almost surely to X is that $\forall\epsilon > 0, \sum_{k=1}^{\infty} \Pr(|X_k - X| > \epsilon) < \infty$. In the specific case of the corollary to Theorem 2, we want to show that for $h = g, G$, $\forall\epsilon > 0, \sum_{k=1}^{\infty} \Pr(|q_{\overline{h},k} - 1| > \epsilon) < \infty$. Fix $\epsilon > 0$. Choose n_0 a positive integer such that $\frac{n_0-1}{n_0+1} \geq 1-\epsilon$ and $\alpha \cdot n_0 > 1$ so that $\frac{\alpha k}{2} \leq \lfloor \alpha k \rfloor$ for $k > n_0$. Then, for all $k \geq n_0$, $\Pr(\{|q_{\overline{h},k} - 1| \geq \epsilon\}) \leq \Pr(G = m \cap \bar{v}_M \leq \mu(n) v_{(2)m}) + \Pr(G = M \cap \bar{v}_m \leq \mu(n) v_{(2)M})$.

For $k \geq n_0$, $m = \lfloor \alpha k \rfloor$, $M = k - m$, we know that $P(G = m \cap \bar{v}_M \leq \mu(n) v_{(2)m}) \leq F\left(\frac{1}{2}\right)^M$ and $P(G = M \cap \bar{v}_m \leq \mu(n) v_{(2)M}) \leq F\left(\frac{1}{2}\right)^m$. We can then write for all $k \geq n_0$ that $P(\{|q_{\bar{h},k} - 1| \geq \epsilon\}) \leq 2F\left(\frac{1}{2}\right)^{\frac{\alpha}{2}k}$. Hence,

$$\sum_{k=n_0}^{\infty} P(|q_{\bar{h},k} - 1| > \epsilon) \leq \sum_{k=n_0}^{\infty} 2F\left(\frac{1}{2}\right)^{\frac{\alpha}{2}k}$$

The latter is the partial sum of a geometric sum with a multiplicative term strictly between 0 and 1. This sum is finite. By the Borel Cantelli lemma, the result is proven. ∎

A.3 Proof of Lemma 1

Lemma 1. *For all distributions F, if all v_i, $i \in m$ and $i \in M$ are i.i.d. with distribution F, then for all n and m, $\theta^*_m \leq \frac{1}{1+\binom{M}{m}} \leq \frac{m}{n}$.*

Proof. Call a realization of n values a *profile* Π, and call a *partition* $\mathcal{P}(\Pi)$ a corresponding minority profile $\mathfrak{m}$ and majority profile $\mathfrak{M}$: $\mathcal{P}(\Pi)=\{\mathfrak{m},\mathfrak{M}\}$[34]. The probability of a profile Π depends on the distribution F, but note that because values are i.i.d., given Π any partition $\mathcal{P}(\Pi)$ is equally likely. Call V_m the sum of realized minority values ($V_m = \sum_{i \in m} v_i$), and similarly for V_M ($V_M = \sum_{j \in M} v_j$). Consider any $\mathcal{P}(\Pi)=\{\mathfrak{m},\mathfrak{M}\}$ such that $V_m > V_M$, supposing that at least one such profile Π and partition $\mathcal{P}(\Pi)$ exist. Now, keeping Π fixed, consider an alternative partition $\mathcal{P}'(\Pi)$ such that the values in the minority profile $\mathfrak{m}$ are reassigned to majority voters. By construction, $V_M > V_m$. The values assigned to the remaining $M - m$ majority voters are chosen freely among all realized values in the original majority profile $\mathfrak{M}$. Thus for any $\mathfrak{m}$, there are $\binom{n-m}{M-m} = \binom{M}{M-m} = \binom{M}{m}$ equally likely partitions $\mathcal{P}'(\Pi)$ such that $V_M > V_m$. But then: $\Pr(V_M > V_m|\Pi) \geq \binom{M}{m} \Pr(V_m > V_M|\Pi)$, with inequality because for given Π we are ignoring partitions $\mathcal{P}''(\Pi)$ such that some of $\mathfrak{m}$ values are associated with minority and some with majority voters and $V_M > V_m$.[35]. Now:

$$\Pr(V_M > V_m) = \int_{\Pi} \Pr(V_M > V_m|\Pi) dG \geq \binom{M}{m} \int_{\Pi} \Pr(V_m > V_M|\Pi) dG = \binom{M}{m} \Pr(V_m > V_M)$$

[34]For clarity: for any Π, there are $\binom{n}{m}$ possible partitions $\mathcal{P}(\Pi)$, and for any partition $\mathcal{P}(\Pi)$ there are $m!M!$ possible permutations of values among the different voters, all keeping $\mathcal{P}(\Pi) = \{\mathfrak{m},\mathfrak{M}\}$ constant.

[35]We are not ignoring those such that $V_m > V_M$ because they are taken into account as different initial partitions $\widetilde{\mathcal{P}}(\Pi)$.

where $G = F^n$ is the joint density of a profile Π. But $\Pr(V_m > V_M) = 1 - \Pr(V_M > V_m)$. Hence: $\Pr(V_m > V_M) \leq \frac{1}{1+\binom{M}{m}}$. To establish that $\frac{1}{1+\binom{M}{m}} \leq \frac{m}{m+M}$, note that it is equivalent to $(m-1)!(M-m)! \leq (M-1)!$, or $\binom{M-1}{m-1} \geq 1$, an inequality that holds for all $m \geq 1$. ∎

A.4 Proof of Proposition 2

Proposition 2. *For all n, m , and F, $\theta_m > \theta^*_m$.*

Proof. We know that if $\overline{v}_g > v_{(2)G}$, the equilibrium in Theorem 1 always applies. If $G = m$ (i.e. $v_n \in m$), m wins with probability $\frac{n+3}{2(n+1)}$; if $G = M$ (i.e. $v_n \in M$), m wins with probability $\frac{n-1}{2(n+1)}$ if $\overline{v}_m < \underline{\rho}\overline{v}_M$, and with some probability $\in (\frac{n-1}{2(n+1)}, \frac{n+3}{2(n+1)})$ otherwise. Hence:

$$\theta_m > \frac{n+3}{2(n+1)} \Pr(G = m \cap \overline{v}_M > v_{(2)m}) + \frac{n-1}{2(n+1)} \Pr(G = M \cap \overline{v}_m > v_{(2)M}) \tag{18}$$

The inequality is strict both because equation 18 sets to $\frac{n-1}{2(n+1)}$ the probability of minority victories whenever $\overline{v}_g > \overline{v}_{(2)G}$ and $G = M$, and because it ignores value realizations such that $\overline{v}_g \in (\mu(n)v_{(2)G}, v_{(2)G})$–the condition in Theorem 1 is satisfied, and the minority wins with positive probability.[36] With i.i.d. value draws:

$$\Pr(G = m \cap \overline{v}_M > \overline{v}_{(2)m}) = \Pr(G = M \cap \overline{v}_m > \overline{v}_{(2)M}) = \frac{mM}{n(n-1)}$$

Thus:

$$\theta_m > \frac{n+3}{2(n+1)} \frac{mM}{n(n-1)} + \frac{n-1}{2(n+1)} \frac{mM}{n(n-1)} = \frac{mM}{n(n-1)}$$

Given Lemma 1, the proposition follows if for all m, n, $\frac{m(n-m)}{n(n-1)} \geq \frac{1}{1+\binom{n-m}{m}}$. Define $f_n(m) = m(n-m)\binom{n-m}{m}$. The inequality then amounts to $f_n(m) \geq n(n-1)$. We first show that for given n, $\forall m \in \{1, \ldots, \frac{n-1}{2}\}$, $f_n(m) \geq min\left(f(2), f_n\left(\frac{n-1}{2}\right)\right)$. For $m \in \{2, \ldots, \frac{n-1}{2}\}$:

$$\frac{f_n(m)}{f_n(m-1)} = \frac{m}{m-1} \frac{n-m}{n-m+1} \frac{(n-2m+1)(n-2m+2)}{(n-m+1)m}$$

Define $g(x) = ln\left(\frac{f_n(x)}{f_n(x-1)}\right)$ for $x > 1$. Then $\forall x > 1$, $g'(x) = -\frac{1}{n-x} - \frac{1}{x-1} + \frac{2}{n-x+1} - \frac{2}{n-2x+1} - \frac{2}{n-2x+2}$. Because $\frac{2}{n-x+1} < \frac{2}{n-2x+1}$ for any positive x, $g'(x) < 0$ for all $x > 1$.

[36] Note that such realizations have positive probability for all F with full support.

Consequently, $\frac{f_n(m)}{f_n(m-1)}$ decreases in m on $\{2, \ldots, \frac{n-1}{2}\}$. Moreover, $\frac{f_n(2)}{f_n(1)} = \frac{(n-2)^2(n-3)}{(n-1)^2} \geq 1$ and $\frac{f_n\left(\frac{n-1}{2}\right)}{f_n\left(\frac{n-3}{2}\right)} = \frac{8(n+1)}{(n-3)(n+3)^2} \leq 1$. Therefore, $f_n(m) \geq min\left(f_n(,2), f_n\left(\frac{n-1}{2}\right)\right)$ for all $m \in \{1, \ldots, \frac{n-1}{2}\}$. Substituting $m = \frac{n-1}{2}$ in $f_n(m)$, we find that $f_n\left(\frac{n-1}{2}\right) \geq n(n-1) \Leftrightarrow n^3 - 7n^2 + 7n - 1 \geq 0$, which holds for all $n > 5$. Substituting $m = 2$ in $n^3 - 8n^2 + 17n - 12 \geq 0$, which holds for all $n > 8$. Therefore, if $n \geq 9$, for all m, F, $\theta_m > \theta^*_m$. For $n \in \{3, 5, 7\}$ we can compute directly the lower bound for θ_m, $\frac{m(n-m)}{n(n-1)}$, and the upper bound for θ^*_m, $\frac{1}{1+\binom{n-m}{m}}$, for $m \in \{1, \ldots, \frac{n-1}{2}\}$ and verify that the result continues to hold. ∎

A.5 Proof of Proposition 4

Proposition 4. *For all F, n, and m, if $E_F(v) > \frac{2(n-m)}{n(n-2m+1)} E_{F,n}(v_{(1)})$ then $W < W_0$.*

Proof. Recall that V_m denotes the sum of realized minority values ($V_m = \sum_{i \in m} v_i$), and V_M the sum of realized majority values ($V_M = \sum_{j \in M} v_j$). For value realizations such that the condition in Theorem 1 is not satisfied, the equilibrium construction selects the majority voting outcome, and thus $(W|\overline{v}_g < \mu v_{(2)G}) = (W_0|\overline{v}_g < \mu v_{(2)G})$. When the value realizations belong to regions A_M ($\underline{\rho}\overline{v}_M > \overline{v}_m > \mu v_{(2)M}$) or A ($\overline{\rho}\overline{v}_M > \overline{v}_m > \underline{\rho}\overline{v}_M$), $\overline{v}_M > \overline{v}_m$, and given $m < M$ and i.i.d. values, it follows that $E[V_M|A_M, A] > E[V_m|A_M, A]$. Thus $(W|\overline{\rho}\overline{v}_M > \overline{v}_m > \mu v_{(2)M}) < (W_0|\overline{\rho}\overline{v}_M > \overline{v}_m > \mu v_{(2)M})$. Hence, for all n and m, a sufficient condition for $W < W_0$ is $E[V_M|A_m] > E[V_m|A_m]$. As a reminder, values in A_m satisfy $\overline{v}_m > \overline{\rho}\overline{v}_M, \overline{v}_M > \mu v_{(2)m}$.

Assume a value distribution F with density f. Denote $h(u) = E_F(v|v \leq u)$ the expectation of a random variable drawn from F conditional on being lower than u. We can write

$$P(\bar{v}_m > \overline{\rho}\bar{v}_M) \cdot E[V_M|\bar{v}_m > \overline{\rho}\bar{v}_M] = \int_{x=0}^{1} \int_{z=0}^{\min\left(\frac{x}{\overline{\rho}},1\right)} [z + (M-1)h(z)] M f(z) F(z)^{M-1} m f(x) F(x)^{m-1} dz dx$$

$$P(\bar{v}_m > \overline{\rho}\bar{v}_M) \cdot E[V_m|\bar{v}_m > \overline{\rho}\bar{v}_M] = \int_{x=0}^{1} \int_{z=0}^{\min\left(\frac{x}{\overline{\rho}},1\right)} [x + (m-1)h(x)] M f(z) F(z)^{M-1} m f(x) F(x)^{m-1} dz dx$$

The first integral can be written as

$$\begin{aligned} P(\bar{v}_m > \overline{\rho}\bar{v}_M) \cdot E[V_M|\bar{v}_m > \overline{\rho}\bar{v}_M] &= \int_{z=0}^{1} [z + (M-1)h(z)] M f(z) F(z)^{M-1} \left[F(x)^m\right]_{\overline{\rho}z}^{1} dz \\ &= \int_{z=0}^{1} [z + (M-1)h(z)] M f(z) F(z)^{M-1} \left[1 - F(\overline{\rho}z)^m\right] dz \end{aligned}$$

Likewise, the second integral yields

$$\begin{aligned}
P(\bar{v}_m > \overline{\rho}\bar{v}_M) \cdot E[V_m|\bar{v}_m > \overline{\rho}\bar{v}_M] &= \int_{x=0}^{\overline{\rho}} [x + (m-1)h(x)] m f(x) F(x)^{m-1} \left[F(z)^M\right]_0^{\frac{x}{\overline{\rho}}} dx \\
&+ \int_{x=\overline{\rho}}^{1} [x + (m-1)h(x)] m f(x) F(x)^{m-1} \left[F(z)^M\right]_0^{1} dx \\
&= \int_{x=0}^{\overline{\rho}} [x + (m-1)h(x)] m f(x) F(x)^{m-1} F\left(\frac{x}{\overline{\rho}}\right)^M dx \\
&+ \int_{x=\overline{\rho}}^{1} [x + (m-1)h(x)] m f(x) F(x)^{m-1} dx
\end{aligned}$$

We can then compare the two quantities. If $P(\bar{v}_m > \overline{\rho}\bar{v}_M) \cdot E[V_M|\bar{v}_m > \overline{\rho}\bar{v}_M] - P(\bar{v}_m > \overline{\rho}\bar{v}_M) \cdot E[V_m|\bar{v}_m > \overline{\rho}\bar{v}_M]$ is positive, then $W < W_0$.

We know that $\overline{\rho} < 1$. Given the definition of A_m, a sufficient condition for the difference to be positive is that it is positive at $\overline{\rho} = 1$, since this raises the lower bound on $barv_m$ (generating a higher V_m in expectation) and decreases the lower bound on $\bar{v}_M$ (generating a lower V_M in expectation).

Substitute $\overline{\rho} = 1$ in the equations above. Then,

$$P(\bar{v}_m > \bar{v}_M) \cdot E[V_M|\bar{v}_m > \bar{v}_M] = \int_{z=0}^{1} [z + (M-1)h(z)] M f(z) F(z)^{M-1} \left[1 - F(z)^m\right] dz \quad (19)$$

and

$$P(\bar{v}_m > \bar{v}_M) \cdot E[V_m|\bar{v}_m > \bar{v}_M] = \int_{x=0}^{1} [x + (m-1)h(x)] m f(x) F(x)^{m-1} F(x)^M dx \quad (20)$$

Denote

$$\Delta = \int_0^1 [x + (M-1)h(x)] f(x) F(x)^{M-1} \left[1 - 2F(x)^m\right] dx$$

Let us write $A_k = \int_0^1 x f(x) F(x)^{k-1} dx$. Substituting $h(x) = \frac{\int_0^x v f(v) dv}{F(x)}$,

$$\Delta = A_M - 2A_n + (M-1)\left(\int_0^1 \left[\int_0^x v f(v) dv\right] f(x) F(x)^{M-2} dx - 2\int_0^1 \left[\int_0^x v f(v) dv\right] f(x) F(x)^{n-2} dx\right)$$

Writing $Ev = E_F(v)$,

$$\int_0^1 \left[\int_0^x vf(v)dv\right] f(x)F(x)^{M-2}dx = \frac{1}{M-1}\left[\int_0^x vf(v)dv \cdot F(x)^{M-1}\right]_0^1 - \frac{1}{M-1}\int_0^1 xf(x)F(x)^{M-1}dx$$
$$= \frac{1}{M-1}\left[Ev - A_M\right]$$
$$\int_0^1 \left[\int_0^x vf(v)dv\right] f(x)F(x)^{n-2}dx = \frac{1}{n-1}\left[\int_0^x vf(v)dv \cdot F(x)^{2(M-1)}\right]_0^1 - \frac{1}{n-1}\int_0^1 xf(x)F(x)^{n-1}dx$$
$$= \frac{1}{n-1}\left[Ev - A_n\right]$$

Hence, we have that $\Delta = -2A_n + Ev - 2\frac{M-1}{n-1}(Ev - A_n)$. Now, let us substract Equation 20 from Equation 19.

$$(19) - (20) = M\Delta + M\int_0^1 [x + (M-1)h(x)]f(x)F(x)^{n-1}dx - \int_{x=0}^1 [x + (m-1)h(x)]mf(x)F(x)^{n-1}dx$$
$$= M\Delta + M \cdot A_n + \frac{M(M-1)}{n-1}(Ev - A_n) - m \cdot A_n - \frac{m(m-1)}{n-1}(Ev - A_n)$$

By simple manipulations, aggregating the terms in Ev and A_n, one finds

$$(19) - (20) > 0 \Leftrightarrow \frac{2M - (n-1)}{2M}Ev \geq A_n$$

From the definition of A_n, and denoting $E(v_{(1)}) = E_{F,n}(v_{(1)})$ we know that $A_n = \frac{1}{n}E(v_{(1)})$. Hence, the condition can be expressed as $n\frac{2M-(n-1)}{2M}Ev \geq E(v_{(n)})$ which is equivalent to the condition given in the statement of the proposition.

∎