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A simple scheme to improve the efficiency of referenda ☆

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ABSTRACT

Referenda are becoming a common tool for public decision-making, and as reliance on direct democracy increases, so does the importance of giving representation to strongly held minority preferences. This paper discusses a very simple scheme that treats everybody symmetrically but gives weight to intense preferences: voters faced with a number of binary proposals are given one regular vote for each proposal plus a single additional bonus vote to cast as desired. Decisions are then taken according to the majority of votes cast. We study the scheme in a number of different models and identify empirically plausible conditions under which ex ante utility increases, relative to simple majority voting.

1. Introduction

In binary decisions – when a proposal can either pass or fail – majority voting has a number of important qualities: it treats all voters symmetrically, it is neutral towards the two alternatives, it reflects accurately changes in preferences in either direction, and it guarantees that voters cannot gain by lying about their preferences. It has, however, one drawback: it fails to account for the intensity of these preferences. Far from being a detail, this one weakness contributes to important practical problems: first of all, the possibility to inflict great harm to the minority; more generally, the blocking of proposals that would increase conventional measures of social welfare, the temptation to recur to log-rolling in committees, the common lack of transparency of political deliberations. In all democratic systems, sophisticated institutions are designed to counter these difficulties. In some cases, however, it may be useful to approach the problem more directly, and ask whether a voting system as simple as majority voting but rewarding intense preferences could be designed for binary decisions.

The functioning of prices in a market offers some inspiration: prices elicit consumers' intensity of preferences by differentiating across goods and functioning in tandem with a budget constraint. The budget constraint plays a central role and suggests an immediate idea: suppose voters were given a stock of votes and asked to allocate them as they see fit over a series of binary proposals, each of which would then be decided on the basis of the majority of votes cast. Would voters be led to cast more votes over those issues to which they attach more importance? And would the final result then be an expected welfare gain, relative to

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simple majority voting, as the probability of winning a vote shifts for each voter from issues of relatively less importance towards issues of relatively more importance? We have proposed a voting system of this type – *storable votes* – in two recent papers that study voting behavior in committees (Casella, 2005; Casella et al., 2006). The simple intuition proves correct: both in theory and in experiments subjects cast more votes when the intensity of their preferences is higher. The efficiency gains are also borne out: both in theory and in the experiments, ex ante utility is typically higher with storable votes (although some counterexamples exist). Hortala-Vallve (2006) has explored a very similar mechanism, independently.

In this paper, we propose to apply this idea to referenda.[1] There are several reasons to do so. First, as tools for policy-making, referenda are becoming both more common and more important, a point made abundantly clear for example by the derailing of the European Union's constitutional treaty in referenda in France and in the Netherlands, and more recently in Ireland.[2] The increased reliance on direct democracy makes the protection of strong preferences, possibly held by minorities, particularly important. Second, referenda are often submitted to voters in bundles—think of the sets of propositions on which voters vote contemporaneously in many US states and European countries.[3] Consider then a voting mechanism where voters are faced with a number of contemporaneous, unrelated referenda, and are asked to cast one vote on each referendum but in addition are given one extra “bonus vote” to cast as desired over one of the different referenda. The value of the bonus vote may but need not be equivalent to that of a regular vote. Each referendum is then decided according to the majority of all votes. Does the simple addition of the bonus vote allow voters to express the intensity of their preferences and increase their ex ante welfare, relative to simple majority voting? This is the question studied in this paper.

We begin by addressing the problem with a simple model where individual valuations are drawn independently from a known distribution, identical across both voters and referenda and symmetrical with respect to the direction of preferences. We find that the answer is positive if the value of the bonus vote is not too large. Intuitively, the bonus vote gives voters the possibility to target the single issue that is most important to them, but at the cost of more uncertainty over the other proposals. The trade-off between the two effects implies that the optimal value of the bonus vote should be related to the expected wedge between the representative voter's highest expected valuation and his mean valuation over all proposals. If such a wedge is small, the value of the bonus vote should be correspondingly small. But the value should not be zero: for all distributions of valuations there is a positive bonus vote value such that ex ante welfare rises, relative to simple majority voting.

After presenting our analysis in the simplest setting, we devote the rest of the paper to relaxing different assumptions and checking the robustness of the first result. We verify that the result continues to hold if the distributions of valuations differ across referenda, as seems plausible. We study whether the result holds when the probability of approval of each referendum is itself a random variable, and find that in this case the conclusion is strengthened: granting a bonus vote always increases ex ante welfare, relative to majority voting, regardless of the bonus vote value.

In all of these cases, we rule out systematic asymmetries between supporters and opponents of each proposal, an assumption that simplifies the analysis greatly and is common in the literature, but limits the role of the bonus vote. Intuitively, the bonus vote improves over majority voting when the preferences of the minority are particularly intense, relative to the majority: its role is exactly to recognize and give weight to possible asymmetries in valuation draws between the two sides of any proposal. When the distributions are assumed to be symmetric, asymmetries can only be occasional sample deviations from the theoretical distributions, bound to disappear in large electorates. Although bonus votes can improve ex ante welfare in all the models discussed above, when the distributions of valuations are assumed to be symmetric, by assumption the per capita quantitative improvement over simple majority voting must become vanishingly small in the limit, as the population approaches infinity. (The same can be said of majority voting over random decision-making).

Recognizing the likely existence of asymmetries in the distributions of valuations is then important, but some restrictions are necessary to keep the model tractable, and their best choice is unclear. In a thorough empirical analysis of more than 800 ballot propositions in California from 1912 through 1989, Matsusaka (1992) identifies an equally split electorate as characteristic of propositions submitted to popular vote (as opposed to being decided by the legislature). Anchoring our model with this observation, we assume that the population is equally split on all proposals, but mean intensity is higher on one side. In this case, the bonus vote is guaranteed to increase ex ante utility if the distribution of valuations on the side with higher mean first-order stochastically dominates the distribution on the opposite side; loosely speaking, if the mass of voters with more intense preferences is larger on the side with higher mean. When this sufficient condition is satisfied, the superiority of the bonus vote over majority voting holds regardless of the exact value of the bonus vote and remains true asymptotically (whereas, with equally split electorates, majority voting again converges to random decision-making). First-order stochastic dominance is a sufficient condition for welfare gains, but our numerical exercises suggest that the result is more general: if the mean intensity of preferences is higher on one side of a proposal, counterexamples where simple majority voting is superior to the bonus vote exist but are not easy to construct.

It is this more general case of asymmetric distributions that better captures the basic intuition for bonus votes. If voters are equally split on a proposal, efficiency demands that the side with the higher intensity of preferences prevails; and if the voters are not equally split, a strongly affected minority should at time prevail over a less affected majority. This is the outcome that bonus

[1] We use the term “referendum” to indicate any proposition decided by popular majority voting, whether initiated by the government (referendum in the proper sense) or by the people (initiative).

[2] Gerber (1999), Matsusaka (2004), the Initiative and Referendum Institute at www.iandrinstitute.org, and the Direct Democracy Institute at www.c2d.unige.ch provide a wealth of information on the history and practice of direct democracy around the world. Referenda are now used in many democracies (in Switzerland, of course, but also in the U.S., the European Union, Australia, and other countries), and their number is rising (in US states, for example, the number of referenda has increased in every decade since 1970, at an average rate of seventy per cent per decade).

[3] In many European countries, the practice of bundling referenda is less common when the stakes are high - a mistake, according to our analysis.

votes can deliver. The conclusion need not involve interpersonal comparisons of utility: in the ex ante evaluation, at a constitutional stage taking place before specific ballots are realized, all voters are identical and the representative voter weighs the probabilities of his yet unrevealed valuations.[4]

But is the need for stronger minority representation a real need in practice? Anecdotal reports abound on the distorting effects of money in direct democracy, and more precisely on the disproportionate power of narrow business interests.[5] Is there room for a voting scheme that is designed to increase further the power of minorities? Perhaps surprisingly, the informed answer seems to be yes. Gerber (1999) and Matsusaka (2004) provide exhaustive empirical analyses of direct democracy in US states, where money spent in referenda campaigns is largest and unlimited. Although their emphasis differs, they both conclude that there is no evidence that business interests are succeeding at manipulating the process in their favor any more than grass-root citizens' groups (or, according to Matsusaka, away from the wishes of the majority). In fact both books isolate the need to protect minorities, stripped of the checks and balances of representative democracy and of the pragmatic recourse to log-rolling, as the most urgent task in improving the process.

The protection of minorities is the heart of the existing voting system that most closely resembles the mechanism described here. *Cumulative voting* applies to a single multi-candidate election and grants each voter a number of votes equal to the number of positions to be filled, with the proviso that the votes can either be spread or cumulated on as few of the candidates as desired. The system has been advocated as an effective protection of minority rights (Guinier, 1994) and has been recommended by the courts as redress to violations of fair representation in local elections (Pildes and Donoghue, 1995; Issacharoff et al., 2001). There is some evidence, theoretical (Cox, 1990), empirical (Bowler et al., 2003), and experimental (Gerber et al., 1998) that cumulative voting does indeed work in the direction intended. The bonus vote scheme discussed in this paper differs because it applies to a series of independent decisions, each of which can either pass or fail, but the intuition inspiring it is similar.

The idea of eliciting preferences by linking independent decisions through a common budget constraint can be exploited quite generally, as shown by Jackson and Sonnenschein (2007). Their paper proposes a specific mechanism that allows individuals to assign different priority to different outcomes while constraining their choices in a tightly specified manner. The mechanism converges to the first best allocation as the number of decisions grows large, but the design of the correct menu of choices offered to the agents is complex, and the informational demands on the planner severe—the first best result comes at the cost of the mechanism's complexity. The recourse to bonus votes in referenda that we discuss in the present paper builds on the same principle but with a somewhat different goal: a mechanism with desirable if not fully optimal properties that is simple enough to be put in practice. It is this simplicity that we particularly value: we propose and study a minor, plausible modification to existing voting practices.

The paper proceeds as follows. Section 2 describes the model; Section 3 establishes the first result and discusses its intuition in the simplest setting, when the distributions of valuations are identical across individuals and proposals and are known to be symmetrical between opponents and supporters of each proposal. Section 4 extends the model to the case where distributions remain symmetrical but differ across proposals. Section 5 studies the bonus vote mechanism when the probability of approval of each proposal is stochastic. Section 6 addresses the case of asymmetric distributions. Section 7 discusses briefly two final points: the effect of granting multiple bonus votes, and the possibility of correlated referenda. Section 8 concludes. All formal proofs can be found in the Appendix.

2. The basic model

A large number n of voters are asked to vote, contemporaneously, on a set of k unrelated proposals (with $k>1$). Each proposal can either pass or fail, and we will refer to the votes as unrelated referenda. Each voter is asked to cast one vote in each referendum, but in addition is given one extra bonus vote. It is natural to think of the bonus vote as equivalent to a regular vote, but we can suppose, more generally, that the bonus vote is worth θ regular votes, with $\theta>0$. For example, imagine regular votes as green, and bonus votes as blue; if $\theta=1/2$, it takes 2 blue votes to counter 1 green vote, and vice versa if $\theta=2$. We constrain θ to be either an integer, or the inverse of an integer; the correct determination of its value is part of the design of the mechanism.

The valuation that voter i attaches to proposal r is summarized by v_{ir}. A negative valuation indicates that i is against the proposal, while a positive valuation indicates that i is in favor. If the referendum is resolved in the voter's preferred direction, the voter's payoff equals the valuation's absolute value, denoted by v_{ir} (in italics); if instead the referendum is resolved in the opposite direction, the voter's payoff is 0. Thus v_{ir} summarizes the intensity of i's preferences, and we will refer to it as *i's intensity*. The voter's objective is to maximize his total payoff from the set of referenda.

Individual valuations are drawn, independently across individuals and across proposals from probability distributions $\mathbf{F}\equiv\{F_r(\mathrm{v}), r=1,\ldots,k\}$ that can vary across proposals but are common knowledge. The distributions $\mathbf{F}$ are symmetrical around 0 (there is no systematic difference between voters who oppose and voters who favor any proposal) and have full support normalized to $[-1,1]$. Each individual knows his own valuation over each proposal, but only the probability distribution of the others' valuations. There is no cost of voting.

[4] The asymmetry of the distribution seems natural when talking informally, but is difficult to justify in analyses based on a single referendum. The approach posits cardinal valuations, but on what basis can one side claim a larger mean valuation than the other? A normalization, a reference criterion, is required. Studying multiple proposals contemporaneously provides such a reference.

[5] See, for example, Broder (2000), with the expressive title *Democracy Derailed*. Opposite views on the promise of direct democracy, held with equal strength, are also common: see for example The Economist, Dec 21, 1996 ("The idea that people should govern themselves can at last mean just that") or The Economist, Jan 23, 2003.

A voter's strategy is the vote cast on each referendum, where by convention a negative vote is a vote against a proposal, and a positive vote is a vote in favor of a proposal. The only constraint on the voter is that he has a single bonus vote. We denote by $\mathbf{X}\equiv\{\pm(1+\theta), \pm 1|\sum_{r=1}^{k}|x_{ir}|=k+\theta$ for all $i\}$ the set of feasible strategies and by $\mathbf{x}_i(\mathbf{v}_i, \mathbf{F})$ the votes cast by individual i, where $\mathbf{v}_i$ is the vector of i's valuations. We restrict attention to symmetrical Bayesian equilibria in undominated strategies where, conditional on their set of valuations, all voters select the same optimal strategy. Because there can be no gain from voting against one's preferences, in these equilibria voters vote sincerely. The only decision is the referendum on which to cast the bonus vote. As intuition suggests and we show in the Appendix, with distributions **F** symmetrical around 0 this choice must be independent of the sign of the voter's valuations.[6] We can simplify our notation, and as long as the distributions **F** are symmetrical, work with distributions $\mathbf{G}\equiv\{G_r(v), r=1,\ldots, k\}$ defined over intensities and support [0, 1].

Call ϕ_r a voter's ex ante probability of casting the bonus vote on referendum r (before observing his valuations), where $\sum_{r=1}^{k}\phi_r=1$. Then:

Lemma 1. *Call p_r the probability that voter i obtains the desired outcome in referendum r when casting a regular vote only, and $p_{\theta r}$ the corresponding probability when adding the bonus vote. Then:*

$$\begin{aligned} p_r &\simeq \frac{1}{2}+\frac{1}{\sqrt{2\pi n\left[1+\phi_r\left(\theta^2+2\theta\right)\right]}} \\ p_{\theta_r} &\simeq \frac{1}{2}+\frac{1+\theta}{\sqrt{2\pi n\left[1+\phi_r\left(\theta^2+2\theta\right)\right]}} \end{aligned} \tag{1}$$

Voter i's optimal strategy is to cast the bonus vote in referendum s if and only if:

$$\frac{v_{is}}{v_{ir}} > \sqrt{\frac{1+\phi_s\left(\theta^2+2\theta\right)}{1+\phi_r\left(\theta^2+2\theta\right)}} \quad \forall r\neq s \tag{2}$$

The probabilities are valid up to an approximation of order $O(n^{-3/2})$ (see the Appendix). Voter i will choose to cast the bonus vote in referendum s over referendum r if and only if $v_{is}p_{\theta s}+v_{ir}p_r>v_{ir}p_{\theta r}+v_{is}p_s$. Substituting Eq. (1), we obtain Eq. (2). If Eq. (2) holds for all r's different from s, then the voter will cast the extra vote on referendum s.

We are now ready to characterize the equilibria, and we begin by studying the case of identical distributions. Later we will extend the analysis to heterogenous distributions and finally to asymmetrical distributions, but it is good to build intuition by considering first the simplest scenario.

3. Identical distributions

When $G_r(v)=G(v)$ for all r, intuition suggests a simple strategy: let each voter cast the bonus vote in the referendum to which he attaches the highest intensity. Indeed, we prove in the Appendix that the following proposition must hold:

Proposition 1. *If $G_r(v)=G(v)$ $\forall r$, then there exists a unique equilibrium where each voter casts the bonus vote in referendum s if and only if $v_{is}\geq v_{ir}\forall r$.*

Both the uniqueness of the equilibrium strategy and its simplicity are strong assets of the mechanism. The immediate response to being allowed to cast a bonus vote is to cast it over the issue that matters most.

To evaluate the potential for welfare gains, we use as criterion ex ante efficiency: the expected utility of a voter before valuations are drawn (or equivalently before being informed of the exact slate of proposals on the ballot). By Proposition 1, the expected share of voters casting their bonus vote is equal in all referenda ($\phi_r=1/k$ $\forall r$), implying, by Eq. (1), that the probability of obtaining the desired outcome depends on whether the bonus vote is cast, but not on the specific referendum: $p_r=p$, $p_{\theta r}=p_\theta$ $\forall_r$. Denote by Ev the expected intensity over any proposal, and by $Ev_{(j)}$ the expected jth order statistics among each individual's k intensities (where therefore $Ev_{(k)}$ is the expected value of each voter's highest intensity). Since voters cast their bonus vote in the referendum associated with the highest intensity, expected ex ante utility EU is given by:

$$\mathrm{EU}=Ev_{(k)}p_\theta+\sum_{j=1}^{k-1}Ev_{(j)}p=k(Ev)p+Ev_{(k)}(p_\theta-p). \tag{3}$$

[6] This observation rules out strategies that seem counterintuitive but not a priori impossible. Consider for example the strategy; $|x_{ir}|=(1+\theta)$ if $v_{ir}<v_{is}$ for all $s\neq r$, and $|x_{ir}|=1$ otherwise (i.e. cast the bonus vote on the referendum that is most opposed, or on the least favored if none is opposed), a strategy where the bonus vote is used disproportionately against all referenda. With symmetric distributions **F** and independent draws across voters, all referenda are expected to fail with probability approaching 1, equal across referenda. The probability of being pivotal is then negligible hut identical across referenda, and the best response strategy for each voter is to cast the bonus vote on the referendum with highest intensity. The suggested strategy cannot be an equilibrium.

Substituting Eq. (1) and $\phi_r = 1/k\ \forall r$, we can write:

$$\mathrm{EU} \simeq kEv\left(\frac{1}{2} + \frac{\sqrt{k}}{\sqrt{2\pi n\left(k+\theta^2+2\theta\right)}}\right) + Ev_{(k)}\left(\frac{\theta\sqrt{k}}{\sqrt{2\pi n\left(k+\theta^2+2\theta\right)}}\right). \tag{4}$$

Our reference is expected ex ante utility with a series of simple majority referenda, which we denote as EW, where, as established in Lemma A.1 in the Appendix:

$$\mathrm{EW} \simeq kEv\left(\frac{1}{2} + \frac{1}{\sqrt{2\pi n}}\right), \tag{5}$$

again ignoring terms of order $O(n^{-3/2})$. Comparing Eqs. (4) and (5), we see that both mechanisms dominate randomness (where each proposal is resolved in either direction with probability 1/2), although both converge to randomness, and to each other, as the population approaches infinity (a point we will discuss in more detail later). Thus a plausible scaling of efficiency is the relative improvement of the two mechanisms over randomness. Calling ER expected utility with random decision-making, we define our measure of welfare improvement as ω, where

$$\omega \equiv \left(\frac{\mathrm{EU}-\mathrm{ER}}{\mathrm{EW}-\mathrm{ER}}\right). \tag{6}$$

We will state that the voting mechanism improves efficiency over a series of simple majority referenda if $\omega > 1$.

Substituting Eqs. (4) and (5) and $\mathrm{ER} = kEv(1/2)$, we derive immediately[7]

$$\omega = \frac{kEv + \theta Ev_{(k)}}{Ev\sqrt{k^2 + k\theta^2 + 2k\theta)}}, \tag{7}$$

which then implies:

Proposition 2. *For any distribution $G(v)$ and any number of referenda $k > 1$, there exists a $\bar{\theta}(G,k) > 0$ such that $\omega > 1$ for all $\theta < \bar{\theta}$.*

The proposition follows immediately from Eq. (7). Indeed, a simple manipulation shows,

$$\omega > 1 \begin{cases} \forall\theta > 0 & \text{if } \left(Ev_{(k)}\right)^2 \geq k(Ev)^2 \\ \forall\theta < \dfrac{2kEv\left(Ev_{(k)} - Ev\right)}{k(Ev)^2 - \left(Ev_{(k)}\right)^2} & \text{if } \left(Ev_{(k)}\right)^2 < k(Ev)^2 \end{cases}. \tag{8}$$

Given a specific distribution, the admissible range of θ values is easily pinned down. Suppose for example that $G(v)$ is the uniform distribution; then $Ev = 1/2$ and $Ev_{(k)} = k/(k+1)$, implying that efficiency improves for all $\theta < 2(k+1)/(k-1)$. If $k = 2$, the constraint is $\theta < 6$: the bonus vote cannot count more than 6 regular votes; if $k = 5$, the constraint is $\theta < 3$, and so forth. Because the ceiling on θ is declining in k, its limit as k approaches infinity provides a sufficient condition for efficiency gains: for any number of referenda, $\theta < 2$ guarantees $\omega > 1$.

In fact we can do more: from Eq. (7) we can derive the optimal θ, the value of the bonus vote that maximizes the efficiency gains, which we denote by θ^*. For arbitrary $G(v)$,[8]

$$\theta^* = \frac{k\left(Ev_{(k)} - Ev\right)}{kEv - Ev_{(k)}}. \tag{9}$$

If $G(v)$ is a uniform distribution, then $\theta^* = 1$ for any value of k: regardless of the number of referenda, the optimal value of the bonus vote is 1—that is, the bonus vote should be equivalent to a regular vote. At $\theta = 1$ and for a uniform $G(v), \omega = \sqrt{k(3+k)}/(1+k)$, always larger than 1, but maximal at $k^* = 3$: given the optimal choice of θ, the number of contemporaneous referenda that maximizes efficiency gains is 3. At these parameter values, the welfare gain relative to simple majority, as defined by ω, is 6%.[9]

[7] Eq. (7) holds for any large n, including in the limit as n approaches infinity.

[8] For arbitrary $G(v)$,

$$Ev_{(k)} = k\int_0^1 v[G(v)]^{k-1}\mathrm{d}G(v),$$

ensuring that the denominator in Eq. (9) is always positive.

[9] We should keep in mind that ω is scaled by randomness. As n approaches infinity, the relative improvement expected from the bonus vote scheme remains 6% but the absolute magnitudes become vanishingly small.

These results, so surprisingly clean, extend easily to a general power distribution, and we summarize them in the following example:

Example 1. *Suppose that $G(v)$ can be parameterized as a power distribution: $G(v)=v^b$, $b>0$. Then, ignoring integer constraints: (i) For all k, $\omega>1$ if $\theta<2/b$. (ii) For all k, $\theta^*=1/b$. (iii) If $\theta=\theta^*$, $k^*=2+1/b$.*

The parameter b determines the shape of the distribution, reducing to the uniform if $b=1$. If $b<1$, $G(v)$ is unimodal at 0, and the mass of voters declines monotonically as intensities become more extreme; with $b>1$, on the contrary, the distribution is unimodal at 1, the upper boundary of the support, and the mass of voters increases with intensity. For a more intuitive understanding of what the distribution implies, suppose for example that voters were asked to rank an issue as "not important," "somewhat important," "important," or "very important," and that these labels corresponded to a partition of the range of possible intensities into 4 intervals of equal size, from [0, 0.25] to [0.75, 1]. With a uniform distribution of intensities, a quarter of the voters would choose each interval; with $b=1/2$, half of the voters would classify the issue as "not important" and about 13% as "very important"; with $b=2$, the percentages become 6% for "not important" and close to 45% for "very important." The parameter b is thus a measure of the saliency of the issue, and the more salient the set of issues, the smaller is the optimal value of the bonus vote: with $b=1/2$, the bonus vote should count as 2 regular votes; with $b=2$ as half, and with $b=3$ as a third.

The ability to state the sufficient condition (i) in the example above is important. Without precise knowledge of the distribution, a policy-maker cannot set the optimal value of the bonus vote, but if the more modest goal of *some* improvement over simple majority is acceptable, this can be achieved by choosing a conservatively small θ. Consider for example setting $\theta=1/2$—then, for all k, efficiency gains are achieved as long as $b<4$. With $b=4$, almost 70% of the voters consider the issue "very important," more than 90% either "important" or "very important" and less than 1% "not important." As long as saliency is not higher, welfare is improved by the bonus vote.

Why is there a ceiling on the acceptable values of the bonus vote? And why does this ceiling depend on the shape of the distribution? Taking θ as given, we can rewrite the condition for efficiency gains (Eq. (8)) as:

$$\omega > 1 \Leftrightarrow \frac{Ev_{(k)}}{Ev} > \frac{k}{\theta}\left(\sqrt{1+\frac{\theta^2+2\theta}{k}}-1\right) > 1 \quad \forall\theta > 0. \tag{10}$$

Condition (10) makes clear that an improvement in efficiency requires a sufficient wedge between the mean intensity and the highest expected intensity draw. The problem is that the introduction of the bonus vote creates noise and redistributes the probability of winning towards the referendum where the bonus vote is utilized but away from the others. Efficiency can increase only if the higher probability of being on the winning side is enjoyed over a decision that really matters to the voter, a decision that matters enough to compensate for the decline in influence in the other referenda. Predictably, the required wedge is increasing in θ: the higher the value of the bonus vote, the larger the noise in the vote distribution and the larger the shift in the probability of winning towards the referendum judged most important. Eq. (1) shows this effect clearly. Similarly, the wedge is increasing in k: the larger is k, the more are the issues over which the probability of winning declines $(k-1)$, and thus again the larger must be the intensity attached to the referendum over which the bonus vote is spent.[10]

For our purposes, the ratio $Ev_{(k)}/Ev$ summarizes all that matters about the distribution of intensities. With a power distribution $Ev=b/(1+b)$, $Ev_{(k)}=bk/(bk+1)$, and $Ev_{(k)}/Ev=(k+bk)/(1+bk)$, an expression that is declining in b. The more salient the issues – the higher b – the smaller the expected difference between the highest draw and the mean intensity, and the smaller must then θ be if Eq. (10) is to be satisfied. Hence the result described above. More generally, given $Ev_{(k)}/Ev$ and k, condition (8) specifies the constraint on θ and Eq. (9) θ's optimal value.[11]

Summarizing, the voting scheme exploits the variation in intensities to ensure that the added noise created by the bonus vote is compensated by a higher probability of winning a decision that really matters. The higher average intensity – the more polarized the society – the higher the variance must be for a given value of the bonus vote, or equivalently, the smaller must be the value of the bonus vote; the lower average intensity, the lower the required variance or equivalently the higher the optimal value of the bonus vote.[12]

4. Heterogeneous distributions

The assumption that intensities are identically distributed over all proposals is, in general, unrealistic: many issues put to referendum are typically of interest only to a small minority – the calendar of the hunting season, the decision to grant landmark status to a building, the details of government procedures – while some on the contrary evoke strong feelings from most voters – divorce in Italy, affirmative action and taxation in California, equal rights for women in Switzerland. [13] Allowing for different

[10] But $Ev_{(k)}$ is also increasing in k. Whether fulfilling Eq. (10) becomes more or less difficult as k increases depends on the distribution.

[11] It was tempting to conjecture a link between the ordering of distributions in terms of the ratio $Ev_{(k)}/Ev$ and first-order stochastic dominance-until Russell Davidson provided a counterexample.

[12] The ratio $Ev_{(k)}/Ev$ depends both on the variance of $G(v)$ and on the mean. A power distribution conflates the two, since both depend on b. (The variance equals $b/[(1+b)^2(2+b)]$ with a maximum at $b=0.62$). A beta distribution is more flexible and isolates the two effects, but does not provide a closed form solution for the kth order statistics. We can nevertheless check conditions (8) or (10) numerically. Suppose for example $\theta=1/2$. Then if $E(v)=1/2$, (10) is satisfied for all k as long as the variance is larger than 0.008 (or equivalently as long as not more than 3/4 of the population are concentrated in the two deciles around the mean). But if the mean is 3/4, the minimum variance rises to 0.02 (or not more than 50% of the population in the two deciles around the mean); if instead the mean is 1/4, the minimum variance falls to 0.002 (or not more than 98% of the population in the two deciles around the mean). The necessary floor on the variance rises as the mean increases.

[13] The distinction is equivalent to Matsusaka's (1992) empirical classification of initiatives into "efficiency" (low salience) and "distributional" (high salience).

distributions makes the problem less transparent, but does not change its logic and in fact increases the expected dispersion in intensities that makes the voting scheme valuable.

The first step is the characterization of the equilibrium – the choice of the referendum on which to cast the bonus vote. Lemma 1 continues to apply, but now voters' bonus votes will not be spread equally over all referenda – the more salient issues will receive a larger share of bonus votes. In equilibrium, ϕ_r, the expected share of voters casting their bonus vote in referendum r, must satisfy

$$\phi_r = \int_0^1 \prod_{s \neq r} G_s(\min(\alpha_{sr} v, 1)) g_r(v) \mathrm{d}v, \tag{11}$$

where

$$\alpha_{sr} \equiv \sqrt{\frac{1+\phi_s\left(\theta^2+2\theta\right)}{1+\phi_r\left(\theta^2+2\theta\right)}} \tag{12}$$

When $G_r(v)=G_s(v)$ $\forall r, s$, as in the previous section, Eqs. (11) and (12) simplify to $\phi_r=1/k$ and $\alpha_{sr}=1$. This is not the case now.

The equilibrium remains unique[14] but is less intuitive than in the case of identical distributions: if a referendum evokes more intense preferences and more voters are expected to cast their bonus vote on that issue, then the impact of the bonus vote will be higher elsewhere. It may be preferable to cast one's bonus vote in a different referendum, even if the intensity is slightly lower. For example, in the case of 2 referenda, $\theta=1$, and power distributions, suppose $b_1=1$ and $b_2=2$. Then $\alpha_{12}=0.89$ – a voter casts the bonus vote on issue 1 as long as $v_1 \geq 0.89\, v_2$ – and the expected shares of bonus votes cast on the two referenda are $\phi_1=0.41$ and $\phi_2=0.59$. If $b_2=4$, the numbers become $\alpha_{12}=0.82$, $\phi_1=0.34$ and $\phi_2=0.66$.

The condition for efficiency gains over simple majority again follows the logic described earlier, but is made less transparent by the need to account for the different distributions and for the factors of proportionality α_{rs}:

$$\omega > 1 \Leftrightarrow \sum_{r=1}^{k} \left[\left(\int_0^1 \prod_{s \neq r} G_s(\min(\alpha_{sr} v, 1)) v g_r(v) \mathrm{d}v\right) \theta \beta_r\right] > \sum_{r=1}^{k} E_r(v)(1-\beta_r), \tag{13}$$

where

$$\beta_r \equiv \frac{1}{\sqrt{1+\phi_r\left(\theta^2+2\theta\right)}}.$$

Condition (13) is analogous to Eq. (10), but because the parameters β_r and α_{sr} differ across distributions and α_{sr} in general differs from 1, it does not reduce to a simple condition on the ratio of the expected highest intensity draw to the mean intensity. Nevertheless, it remains possible to state:

Proposition 3a. *For any set of distributions* **G** *and for any number of referenda $k>1$, there exists a $\tilde{\theta}(G,k)>0$ such that $\omega>1$ for all $\theta<\tilde{\theta}$.*

The proposition is proved in the Appendix. It states that the result we established earlier in the case of identical distributions is in fact more general, and continues to apply with heterogeneous distributions.

In practical applications, two concerns remain. The first is that calculating the equilibrium factors of proportionality α_{rs} is not easy. How well would voters fare if they followed the plausible rule of thumb of casting the bonus vote on the proposal with highest intensity? It seems wise to make sure that the desirable properties of the mechanism are robust to the most likely off-equilibrium behavior. In fact, Proposition 3a extends immediately to this case:

Proposition 3b. *Suppose voters set $\alpha_{sr}=1 \forall s, r$.* Proposition 3a *continues to hold. (See the* Appendix).

The second concern was voiced earlier. If the planner is not fully informed on the shape of the distributions, or if the value of θ is to be chosen once and for all, for example in a constitutional setting, can we identify sufficient conditions on θ that ensure efficiency gains for a large range of distributions? The answer is complicated by the factors α_{rs} and thus by the lack of a simple closed-form solution even when we specialize the distributions to simple functional forms. However, in our reference example of power distributions and in the "rule-of-thumb" case where voters cast the bonus vote on the highest valuation proposal, we obtain an interesting result:

Example 2. *Suppose $G_r(v)=v^{b_r}$, $b_r>0$ $\forall_r$, and set $\alpha_{sr}=1$ $\forall s,r$. Call $b_k \equiv \max\{b_r\}$. Then for all $k>1$, $\omega>1$ if $\theta \leq 1/b_k$.*

The example is proved in the Appendix. As in the case of identical distributions, we can derive a simple sufficient condition ensuring welfare gains: the value of the bonus vote can be safely set on the basis of the distribution of intensities in the most

[14] Consider an equilibrium $\{\phi_r'\}$. Posit a second equilibrium $\{\phi_r''\}$. Suppose $\phi_s''>\phi_s'$. Then, given Eqs. (11) and (12) there must exist at least one issue r such that $\phi_r''/\phi_r'>\phi_s''/\phi_s'$. Call z the issue such that ϕ_z''/ϕ_z' is maximum. Then $\alpha_{rz}(\phi_z'',\phi_r'')<\alpha_{rz}(\phi_z',\phi_r')$ $\forall r \neq z$, and by Eq. (11) $\phi_z''<\phi_z'$ establishing a contradiction. Reversing the signs, the identical argument can be used to show that there cannot be an equilibrium with $\phi_s''<\phi_s'$.

strongly felt of the issues under consideration. If we return to our previous discussion and partition the support of intensities into four equal size intervals, setting $\theta=1/4$ or $1/5$ would seem a prudent policy.[15] Intuitively, we expect the condition to be stronger than needed: the heterogeneity of the distributions should help in providing the spread in expected intensities that underlies the voting scheme's efficiency gains. Indeed, in all our numerical exercises with power distributions we achieved welfare gains by setting $\theta \leq k/\sum_{r=1}^{k} b_r$, the inverse of the mean b parameter, a looser constraint than $1/b_k$.[16]

This section allows us to conclude that the properties of the voting scheme, so transparent in the simple case of identical distributions, extend to the more plausible scenario of heterogeneous distributions. Having established the result in our basic model, we can now study its robustness when we relax the model's most restrictive assumptions.

5. Stochastic probability of approval

The assumption that the distributions of valuations **F** are symmetric around 0 implies that the medians of the distributions are known and equal to 0. Because voters' valuations are independent draws from these distributions, the model is equivalent to one where each voter supports each proposal with known probability equal to 1/2.

As mentioned in the Introduction, studying all ballot propositions in California in the period 1912–89, Matsusaka (1992) concludes that the legislature consistently defers to popular vote issues that ex ante appear particularly contested, i.e. where the electorate is approximately equally split. Thus empirically, the assumption of an equally split electorate in referenda is not implausible, but how important is it for our results? As remarked in the literature, a more general assumption is that for each referendum the probability of approval is not known ex ante.[17] This is the case we study in this section.

Suppose that ex ante each voter had a probability ψ_r of being in favor of proposal r, and $1-\psi_r$ of being against. The probability ψ_r is distributed according to some distribution H_ψ defined over the support [0, 1], and symmetric around 1/2: although it is recognized that the probability of approval is uncertain, there is no expected bias in favor or against the referendum. Each realized ψ_r is an independent draw from H_ψ. Conditional on being in favor or against, the distribution of intensities continues to be described by $G_r(v)$, defined over support [0, 1] and thus equal for voters in favor and voters against the proposal. It seems correct to assume that the popularity of a proposal has no implication for the relative intensity of preferences of supporters and opponents: there is no systematic bias in the intensity of preferences of the minority, relative to the majority. Finally, for simplicity we maintain the assumption that $G_r(v)=G(v)$ for all r.

In the absence of systematic differences across referenda, in equilibrium voters continue to cast their bonus vote in the referendum to which they attach the highest intensity: the stochastic probability of approval does not affect the equilibrium strategy. But it does affect the welfare comparison, strengthening the argument in favor of the bonus vote. Defining $\omega_s \equiv (\mathrm{EU}_s-\mathrm{ER})/(\mathrm{EW}_s-\mathrm{ER})$, where the subscript identifies the model with stochastic approval, we show in the Appendix that:

$$\omega_s = \frac{k(Ev)+\theta Ev_{(k)}}{(Ev)(k+\theta)} \tag{14}$$

and thus the following result holds:

Proposition 4. *For all distributions $H_\psi(\psi)$ and $G(v)$, and for all $k>1$, $\omega_s>1$ for all $\theta>0$.*

If the probability of approval is uncertain, ex ante utility increases, relative to simple majority, *for all values* of the bonus vote. Concerns about identifying the correct ceiling on θ do not apply to this model. The intuitive reason follows from our previous discussion. As we saw, bonus votes increase the variability of the total votes cast in each referendum, and thus reduce the probability of being pivotal in referenda in which the voter does not cast his bonus vote, relative to simple majority voting. This effect continues to exist when the probability of approval of each referendum is stochastic, but now has a second, positive implication: the increase in variability works to reduce the impact of non-balanced expected total votes on the probability of being pivotal. The net result is that the decline in the probability of being pivotal when the bonus vote is not cast is reduced, and reduced sufficiently to guarantee that the overall effect of the bonus vote is an increase in expected welfare, relative to simple majority, for any positive value of θ.

The analysis in the Appendix yields several additional observations. First, as established in the literature, the probability of being pivotal is of order $1/n$, a result that holds true both with majority voting and with the bonus vote scheme.[18] Second, what complicates the analysis with the bonus vote is not the stochastic nature of ψ_r but the feedback between the expected distribution of the votes in each referendum and the voters' best response strategy. The modelling assumptions made in this section, and in particular the lack of systematic differences across referenda ($G_r(v)=G(v)$ for all r), and the symmetry of H_ψ, allow us to pin down the equilibrium strategy in a tractable manner. More general formulations would be more difficult to solve, although, in line with the previous section, we see no obvious reason why the conclusions should change. Third, the sharp welfare result does depend on

[15] With $b=5$, more than 3/4 of all voters consider the issue "very important," 97% consider it either "important" or "very important," and less than 1 in a thousand "not important."

[16] This was true whether we looked at the equilibrium or at the $\alpha_{sr}=1$ case. With $k=2$, efficiency gains in the "rule-of-thumb" scenario are sufficient for efficiency gains in equilibrium, but not with $k>2$.

[17] See for example Good and Mayer (1975), Margolis (1977) and Chamberlain and Rothschild (1981). Gelman et al. (2002) discuss the implications of a number of alternative models.

[18] With the bonus vote scheme, the probability of being pivotal is $\frac{1}{2(1+\theta/k)}\frac{1}{n}h_\psi(1/2)$ when the bonus vote is not cast, and $\frac{1+\theta}{2(1+\theta/k)}\frac{1}{n}h_\psi(1/2)$ when it is. If $\theta=0$, we reproduce the standard result for simple majority voting.

one assumption: the lack of positive correlation between the volume of approval ψ_r and the expected intensity of preferences Ev. Alternative models are possible. For example, we can think of the stochastic probability of approval as a stochastic shift in the centers of the distributions **F**: in this case, $F_r(v)$ has support $[-(1-c_r), 1+c_r]$ with c_r distributed according to some H_c over $[-1, 1]$ and $\psi_r=(1+c_r)/2$. Now the expected intensity of preferences depends on their direction, and because the whole distribution of valuations moves to the right when approval for the referendum is higher (and to the left when it is lower), the minority is constrained to have less intense preferences than the majority by assumption. Indeed, the smaller the minority the weaker its preferences. Bonus votes, meant to differentiate between popular support and intensity of preferences, would be less valuable in this model. We conjecture that they would improve expected welfare over simple majority only if valuations are sufficiently concentrated around the center of the distribution, de facto reducing the correlation between volume and intensity of support. This observation can be important in specific practical applications, but on the whole we see no a priori reason why the minority should systematically have weaker preferences than the majority.

6. Asymmetrical distributions of valuations

For all the subtleties of the different models, the intuitive reason why bonus votes can be valuable is straightforward: they give some voice to minority preferences when these are particularly intense, relative to majority preferences. In other words, bonus votes recognize possible asymmetries in intensities that majority voting ignores.

When we work with symmetrical distributions of valuations, as we have done so far, we make that task particularly hard: bonus votes can then only reward occasional empirical asymmetries, sample deviations from the theoretical distribution whose importance must disappear as the population becomes large. This is why the absolute welfare improvement over simple majority disappears in the limit, as we remarked earlier. The observation is almost obvious: if we constrain the mean and the median of the distribution of valuations to coincide (or more generally to have the same sign), simple majority must be asymptotically efficient; it is only when we allow the mean and the median to differ that bonus votes can play a more substantial role.[19]

To study the problem in the simplest setting, suppose that the distributions of valuations are identical over all proposals, but now for each proposal call $P(v)$ the distribution of intensities of voters in favor, and $C(v)$ the distribution of intensities of voters against the proposal. The two distributions have different means: for concreteness, suppose $E_P(v)>E_C(v)$, implying that in each referendum the mean valuation over the whole electorate is positive. We assign higher mean intensity to the "pro" side with no loss of generality—which side has higher mean is irrelevant and we could trivially generalize the model to allow the side with higher mean to change across proposals.

We go back to our original assumption that the probability of support of a referendum – or equivalently the median valuation over the whole electorate – is fixed: this section studies the scope for welfare gains whose absolute size does not disappear asymptotically, and the welfare comparison to simple majority will not depend on relative probabilities of being pivotal.[20] Where the median is determines the asymptotic welfare properties of majority voting: with $E_P(v)>E_C(v)$, majority voting is asymptotically efficient if the median is positive, inefficient if it is negative, and equivalent to randomness if the median is at 0.[21] Invoking again Matsusaka (1992)'s empirical results, suppose that the median is at 0: both $P(v)$ and $C(v)$ have full support [0, l], and $P(1)=C(1)=1$. All valuations are independent, across voters and proposals.

The asymptotic properties of the bonus votes scheme depend on the shapes of the $P(v)$ and $C(v)$ distributions, mediated by the equilibrium strategy. Once again, the equilibrium strategy is pinned down by the requirement that the impact of the bonus vote must be equalized across referenda and requires voters to cast their bonus vote on the referendum felt with highest intensity.[22] The welfare properties then depend on the probability that the sign of a voter's highest intensity draw equals the sign of the mean valuation in the population in that referendum, i.e. is positive in this example. Call ϕ_{rP} the probability of casting the bonus vote in favor of referendum r, where, given the equilibrium strategy, $\phi_{rP}=\phi_P=\left(\frac{1}{2}\right)^{k-1}\left[\sum_{s=0}^{k-1}\left(\binom{k-1}{s}\int_0^1 C(v)^{k-1-s}P(v)^s p(v)\mathrm{d}v\right]\right.$ for all r. Similarly, call ϕ_{rC} the probability of casting the bonus vote against referendum r, where $\phi_{rC}=\phi_C=\left(\frac{1}{2}\right)^{k-1}\left[\sum_{s=0}^{k-1}\left(\binom{k-1}{s}\int_0^1 C(v)^{k-1-s}P(v)^s c(v)\mathrm{d}v\right]\right.$ for all r. Consider a sequence of bonus votes referenda indexed by the size of the population n, and similarly index our welfare criteria. Then:

Proposition 5. *For any $\theta>0$ and $k>1$, if $\phi_P>\phi_C$ then as $n\to\infty$, $\mathrm{EU}_n/\mathrm{EW}_n\to 1+[E_p(v)-E_C(v)]/[E_P(v)+E_C(v)]>1$, and $\omega_n\to\infty$.* (The proof is in the Appendix).

[19] Ledyard and Palfrey (2002) exploit this logic to design an asymptotically efficient voting referendum: given the distribution of valuations, the critical threshold for approval must be fixed at the level that makes a representative voter ex ante indifferent over the outcome of the referendum. Because the empirical average valuation will converge to the theoretical mean, in a large population the referendum will pass whenever the mean net benefit is positive, and thus will deliver the efficient outcome. With a distribution symmetrical around 0, the threshold corresponds to 50%. (This also implies the asymptotic efficiency of random decision-making). More generally, the asymptotically efficient threshold depends on the distribution. The idea is simple and clever, but setting different thresholds for different decisions seems quite delicate in practice.

[20] With asymmetries and known probability of approval for each referendum, a voter's probability of being pivotal approaches zero at rate e^{-n}, both with simple majority and with bonus votes. We take the willingness to vote as a given.

[21] If the probability of approval ψ_r is random, the welfare results can be rephrased in terms of the median of the H_ψ distribution.

[22] Even in the presence of asymmetries between opponents and supporters of each referendum, the equilibrium is generically unique - see the proof in Casella and Gelman (2005).

As long as $\phi_P > \phi_C$, the probability that a referendum passes converges asymptotically to 1 for any positive θ, as opposed to approaching 1/2 in the case of simple majority. Bonus votes shift the outcome in the direction of the mean, and hence increase efficiency. As the size of the population approaches infinity, majority voting approaches randomness, but bonus votes do not, and the absolute difference in ex ante utility between the two voting mechanisms does not disappear: relative to randomness, the welfare gain associated with bonus votes grows arbitrarily large.

Whether the condition $\phi_P > \phi_C$ is satisfied depends on the shape of the distributions. Given on average higher positive valuations, the condition seems plausible, but guaranteeing it requires imposing further restrictions on the distributions. For example, the definitions of ϕ_P and ϕ_C imply immediately that a sufficient condition is first-order stochastic dominance: if $P(v)$ first-order stochastically dominates $C(v)$, then $\phi_P > \phi_C$, and Proposition 5 follows. First-order stochastic dominance is satisfied by the power distribution we have used as recurring example:

Example 3. *Suppose that both $P(v)$ and $C(v)$ are power distributions with parameters b_p and b_c, where $b_p > b_c$. Then for any $\theta > 0$ and $k > 1$, as $n \to \infty$, $EU_n/EW_n \to 1 + (b_p - b_c)/(b_p + b_c + 2b_p b_c) > 1$ and $\omega_n \to \infty$.*

With first-order stochastic dominance, the probability mass of favorable valuations is concentrated towards higher intensities than is the case for negative valuations, and bonus votes are correspondingly concentrated on favorable votes. To see what first-order stochastic dominance implies in practice, suppose once again that the public's intensity of preferences at best can be identified through a partition of the support of intensities into four equally sized intervals. Consider a referendum where proponents on average have more intense preferences than opponents. First-order stochastic dominance requires some monotonicity in the manner in which voters on the two sides are distributed in the four intervals. Among those judging the proposal "very important" the majority should be proponents, and similarly among those considering it either "very important" or "important"; among those judging the proposal "not important" the majority should be opponents, and similarly among those considering it either "not important" or "somewhat important."

First-order stochastic dominance guarantees that the bonus vote scheme asymptotically dominates simple majority, but is stronger than needed. In a series of numerical simulations, we have studied the question when $P(v)$ and $C(v)$ are beta distributions, with parameters constrained to ensure $E_P(v) > E_C(v)$, but otherwise free and such that first-order stochastic dominance in general does not hold. We have found that violating the condition $\phi_P > \phi_C$ is possible but not easy: in our simulations it requires a number of simultaneous referenda k not too small, and distributions of intensities $P(v)$ and $C(v)$ such that supporters' intensities are concentrated around the mean $E_P(v)$, while opponents' intensities are dispersed and bimodal at 0 and 1. With probability increasing in k, it is then possible for the bonus votes to be used predominantly by opponents, i.e. the side with lower mean intensity (the larger the number of draws, the higher the probability that the highest draw will come from the more dispersed distribution). But the range of parameter values for which this occurs is small, and intuition suggests that it should be smaller still if the distributions differ across referenda.[23]

7. Discussion

7.1. Multiple bonus votes

We have assumed so far that voters are granted a single bonus vote. But would granting multiple bonus votes be preferable? In this section we discuss why, on balance, we believe that the answer is negative. There are two possibilities: either the multiple bonus votes can be cumulated freely by the voters, or they cannot. We discuss the two options in turn.

7.1.1. Cumulative bonus votes

When bonus votes can be cumulated, the strategic problem faced by the voters becomes much more complex because the strategy space is much richer: any combination of bonus votes over the different referenda should be considered. From a practical point of view, the complexity is a drawback, but could be justified if the welfare gains were substantial. However, as the following example shows, the problem is that the equilibrium with multiple bonus votes can in fact be *identical* to the equilibrium with a single bonus vote: there are plausible scenarios such that if the bonus votes can be cumulated, in equilibrium they are, and are all cast by voters on their single highest valuation. In this subsection, we concentrate on "responsive equilibria", i.e. equilibria where voters' strategies depend on intensities, and not on the labels identifying the different referenda.

Example 4. *Suppose all distributions of valuations* **F** *are symmetrical, and $k > 2$. Voters are granted $(k-1)$ bonus votes, each of value equivalent to a regular vote. Then there is a unique responsive equilibrium where all voters cumulate all their bonus votes on their highest intensity referendum.* (The proof is in the Appendix).

It could be objected that bonus votes equivalent to regular votes are too blunt an instrument: the equilibrium described in the example is identical to the equilibrium with a single bonus vote of value $(k-1)$, possibly quite large. Are bonus votes still going to be

[23] The details of the simulations are reported in Casella and Gelman (2005). There is a trade-off involved in the choice of k: the higher is k the larger is $(EU - EW)$ if $\phi_P > \phi_C$; but if $P(v)$ does not first-order stochastically dominate $C(v)$, the higher is k, the larger the range of distributions for which $\phi_P < \phi_C$. Thus the optimal k depends on the precision of the information on the shape of the distributions. In our simulations, the condition $\phi_P > \phi_C$ was always satisfied if $k = 2$.

cumulated when they are worth less than the regular vote? The answer to this question is less straightforward because the equilibrium then depends both on the exact value and on the exact number of the bonus votes. But the previous result may continue to hold. For example:

Example 5. *Suppose all distributions of valuations* **F** *are symmetrical, and $k>2$. Voters are granted $(k-1)$ bonus votes, each of value $\vartheta=1/k$. Then there is a responsive equilibrium where all voters cumulate all their bonus votes on their highest intensity referendum.* (The proof is in the Appendix).

Even with fractional bonus votes the equilibrium may be identical to the equilibrium with a single bonus vote.[24] Our conclusion then is that multiple bonus votes complicate the problem both for the voters and for the planner, and may at the end be indistinguishable from the simple scheme with a single bonus vote. All considered, from a practical point of view, a single bonus vote seems to us a preferable mechanism.

7.1.2. Non-cumulative bonus votes

Alternatively, there could be multiple bonus votes that cannot be cumulated. The mechanism would then resemble closely the one proposed by Jackson and Sonnenschein (2007). As described earlier, the Jackson and Sonnenschein mechanism asks each individual to announce the valuations attached to a series of unrelated decisions, with the menu of allowed announcements constrained to mimic the actual distribution of valuations. As the number of decisions becomes large, the distribution of valuations can be reproduced more and more finely, and the mechanism approaches first best efficiency asymptotically.[25] Suppose for example that the distribution of valuations is uniform. Then agents are simply asked to rank the decisions: the ranking can be read as fitting the same percentage of decisions into any equal subset of the distribution. In our setting, with k binary proposals the mechanism can be implemented through k referenda where each voter is endowed with a series of $k-1$ non-cumulative bonus votes of values 1, 2, …, $k-1$. As k becomes large, the mechanism approaches full efficiency. If the distribution is not uniform, the mechanism does not reduce to a simple ordinal ranking, and, correspondingly, the optimal values of the bonus votes will in general be less intuitive.

The theoretical result established by Jackson and Sonnenschein is a limit result and requires the number of decisions to be large. It is natural to ask how well their mechanism performs when the number of decisions is small, and in particular how it compares to granting a single bonus vote. A full answer is beyond the scope of this paper, but an example can be instructive. As in Jackson and Sonnenschein, suppose that the distributions **F** are identical for all proposals ($F_r=F\ \forall r$), and suppose also that they are symmetrical around 0. Each voter is endowed with k bonus votes of values ϑ_s, with $s\in\{1,\dots,k\}$ and $\vartheta_1<\vartheta_2<..<\vartheta_k$, that cannot be cumulated. Valuations draws are independent both across proposals and across voters.

In the setting we are describing, there is an equilibrium where the highest bonus vote is cast on the referendum with highest intensity, and so on in declining order. The gain in ex ante utility relative to random decision-making is given by:

$$\mathrm{EU}_{\mathrm{nc}}-\mathrm{ER}=\frac{k\mathrm{E}v+\sum_{s=1}^{k}\vartheta_s Ev_{(s)}}{\sqrt{2\pi n\left[1+1/k\left(\sum_{s=1}^{k}\vartheta_s^2+2\vartheta_s\right)\right.}} \tag{15}$$

where the subscript nc indicates the non-cumulative bonus votes case, and $Ev_{(s)}$ denotes the sth order statistics. Because only relative values of the votes matter, we can set $\vartheta_1=0$: the vote cast on the referendum with lowest intensity – the regular vote alone – is the numeraire. The optimal values of the remaining $k-1$ bonus votes, denoted by ϑ_s^*, are chosen by the planner and depend on the distribution of intensities.

Example 6. *Suppose that $G(v)$ is Uniform. Then $\vartheta_s^*=s-1$, $s\in\{1,\dots,k\}$, for all k. At $\vartheta_s=\vartheta_s^*$ for all s, the welfare criterion ω_{nc} is monotonically increasing in k. It equals 1.08 at $k=3$, 1.11 at $k=5$, and converges to $2/\sqrt{3}=1.15$ as k approaches infinity. The comparison to a single bonus vote depends on k. Recall that with a single bonus vote $\theta^*=1$ for all k. If $\theta=1$, $\omega=1.06$ at $k=3$ (where it is maximized), and falls monotonically for larger k, reaching $\omega=1.05$ at $k=5$, and $\omega=1.01$ at $k=50$.*

In this example, multiple non-cumulative bonus votes are valuable, and their welfare improving potential is confirmed even when the number of decisions is small. In fact, a majority of the welfare gains can be reaped with little bundling of referenda—i.e. at values of k equal to 3 or 4, an observation also made by Jackson and Sonnenschein in their numerical simulations. The implication is that a single bonus vote is inferior, but can be reasonably effective if practical considerations constrain k to be small, or if a larger number of referenda can be costlessly divided into several bundles.

We conclude with two further observations. First, in more general cases, identifying the optimal values of the non-cumulative bonus votes is less straightforward and the values themselves are less intuitive. For example, if $G(v)=v^b$, $b=4$ and $k=4$, $\vartheta_2^*=1/4$, $\vartheta_3^*=13/32$ and $\vartheta_4^*=67/128$ (we have been unable to find a simple closed-form solution for ϑ_s^* with arbitrary k). Second, if the

[24] In general, it need not be: we have not shown that the equilibrium in Example 5 is unique, and in fact with k (as opposed to $k-1$) bonus votes of value $\vartheta=1/k$, cumulating all bonus votes is not an equilibrium. We owe this observation, as well as its proof in the Appendix, to an anonymous referee. With k bonus votes, cumulating them all remains an equilibrium if $\vartheta=1$.

[25] The result requires that the valuations draws be independent across individuals and decisions, and that the distributions of valuations be identical over all decisions. We return to this latter point below.

distributions of valuations are less well-behaved – if the probability of approval of different proposals is stochastic, or if the distributions are asymmetrical or heterogenous – characterizing the equilibrium strategies becomes much more challenging. But these are exactly the cases where the potential impact of bonus votes is both more robust and more important. It may well be that in such cases referenda with multiple non-cumulative bonus votes cannot implement the Jackson and Sonnenschein mechanism. In fact, if the distributions of valuations are heterogenous, we are outside the scope of the Jackson and Sonnenschein mechanism and cannot invoke the mechanism's efficiency results, even when equilibrium play can be pinned down.

7.2. Related referenda

We have maintained throughout the paper that the referenda are unrelated: in all of the analysis, each voter's valuations were assumed to be independent across referenda. Strictly speaking, the menu of referenda is part of the design of the mechanism, and we could demand of the planner that the referenda be unrelated. In practice, however, the assumption is likely to be violated. Does it matter? If a voter's utility is not separable in the referenda's valuations, for example if preferences on a specific referendum depend on the outcome of a different one, then the correct model is not one of k binary decisions, but of a single k-dimensional choice, among 2^k possible alternatives. This is a more difficult problem, lending itself to the possible pathologies identified by voting theory.[26] If the assumption of separable utility can be maintained, however, we can address the question within the model we have used so far.

Suppose that each voter's valuations over the k referenda are drawn from a multivariate distribution $F_i(\mathrm{v}_1,\ldots,\mathrm{v}_k)$ which we assume identical across voters: $F_i = F$. Valuations are statistically dependent across referenda, but are independent across voters, and the referenda are held simultaneously. In this case, the previous analysis continues to apply but needs to be rephrased in terms of the marginal (unconditional) distributions of the valuation in each referendum. The difficulty is not the dependence among each voter's valuations per se, but the possible heterogeneity of the marginal distributions. Without more structure on the pattern of dependence, and thus on the distributions' heterogeneity, characterizing the equilibrium strategy is very difficult.

If the marginal distributions are not heterogeneous, however, the analysis is unchanged. More precisely, suppose that the distribution $F(\mathrm{v}_1,\ldots,\mathrm{v}_k)$ is exchangeable, i.e. is invariant to permutations of the indexes of the referenda. Then, although the valuations are statistically dependent, the model is fundamentally symmetrical: ex ante there is nothing to distinguish one referendum from the others. The condition is restrictive, but it is not hard to think of scenarios that satisfy it. Consider the following example, where we make a distinction between dependence in the *direction* of preferences and dependence in the *intensity* of preferences. Suppose that the k referenda concern a single general topic and the direction of each voter's preferences is perfectly correlated among them—if the voter is in favor of one referendum, then he is in favor of all.[27] Intensities, however, are drawn independently across referenda according to some distribution $G_r(v)$, regardless of the sign of the preferences—each individual referendum may be considered by the voter trivial or important, and knowing the intensity over one of them does not help predict the voter's intensity over a different one. If $G_r(v) = G(v)$, then the distribution F is exchangeable. In this example we have assumed perfect correlation in the direction of preferences and zero correlation in the intensity of preferences, but all that is required is that both types of correlation be symmetrical across referenda.

If the distribution $F(\mathrm{v}_1,\ldots,\mathrm{v}_k)$ is exchangeable, the marginal distributions of valuations are identical across referenda, and in equilibrium each voter casts the bonus vote on the referendum with highest intensity. The welfare conclusions depend of the shapes of these distributions, but follow the same logic discussed in the rest of the paper. There are two caveats: first, we are ruling out Bayesian updating on the part of the individual voter that, on the basis of the voter's own valuations, may result in heterogeneous posterior distributions. Second, if the intensities are not independent, the wedge between the mean and the expected highest intensity will be reduced, if the correlation is positive, or increased, if it is negative, affecting the potential for welfare gains.

8. Conclusions

This paper has discussed an easy scheme to improve the efficiency of referenda: hold several referenda together and grant voters, in addition to their regular votes, a single special vote that can be allocated freely among the different referenda. By casting this “bonus vote” on the one issue to which each voter attaches most importance, voters shift the probability of obtaining their preferred outcome away from issues they consider low priorities and towards the one they care most about. Under plausible scenarios, the result is an expected efficiency gain relative to simple majority.

We started our analysis by the simplest cases: scenarios where the distributions of valuations are symmetrical between proposers and opponents of each referendum. In these models, there is always a positive value of the bonus vote for which the scheme ex ante dominates simple majority voting. However, the value of the bonus vote should be not too high, and the quantitative improvement over majority voting disappears asymptotically for very large populations (as indeed equally disappears majority's improvement over random decision-making), the inevitable result of assuming symmetric distributions. The advantages of the bonus vote scheme become more robust when asymmetries in distributions are allowed to enter the model. If the probability of support for each referendum is stochastic, the model remains ex ante symmetrical but allows for ex post differences in the mass of voters on the two sides of a referendum. In this case, the bonus vote scheme ex ante dominates simple majority for *all* values of the bonus vote, and the earlier concern about setting correctly the value of the bonus vote disappears. However, it remains true that the magnitude of the expected welfare gains vanishes for very large populations. Finally, we allow for ex ante

[26] See for example Brams et al. (1998).

[27] More generally, all signs are perfectly predictable, once one of them is known.

asymmetries in the distributions of preferences: we suppose that while voters are equally divided between supporters and opponents of each referendum, one side has preferences of higher mean intensity than the other. Under a plausible regularity condition on the distributions of preferences, again the bonus vote scheme ex ante dominates simple majority for all positive values of the bonus vote, but now in addition the quantitative improvement does not disappear asymptotically.

Bonus votes are a simple mechanism allowing some expression of voters' intensity of preferences. They recall cumulative voting—an existing voting scheme employed in multi-candidate elections with the expressed goal of protecting minority interests. The protection of minorities built into these mechanisms is a particularly important objective as recourse to direct democracy increases. In fact the need to safeguard minorities, and in particular minorities with little access to financial resources, is the single point of agreement in the often heated debate on initiatives and referenda (for example Matsusaka, 2004 and Gerber, 1999).[28] The important objective is designing voting mechanisms that increase minority representation without aggregate efficiency losses, and this is why in this paper we have insisted on the pure efficiency properties of bonus votes.[29]

The paper suggests several directions for future research. In addition to the points raised in the text, one important question the paper ignores is the composition of the agenda. In the model we have studied, the slate of referenda is exogenous. We believe this is the correct starting point: modeling agenda-formation processes is famously controversial, and in our case requires identifying groups with common interests, taking a stance on the correlations of the group members' valuations across different issues and on the forces holding the groups together when voting is completely anonymous. From a technical point of view, it implies renouncing the assumption of independence across voters and thus the power of the limit theorems we have exploited repeatedly. Intuitively, the final outcome seems difficult to predict: bonus votes may increase the incentive to manipulate the agenda, but also the ability to resist the manipulation. We leave serious work on this issue for the future.

A second question left unaddressed is the possible role of voting costs.[30] If voting costs are explicitly considered, do bonus votes still lead to improvements in welfare? The question is relevant because in the presence of voting costs individuals close to indifference should prefer to abstain, and thus voting costs also work to increase the representation of voters with higher intensity of preferences. A rigorous comparison between majority voting and the bonus vote mechanism in the presence of voting costs demands a full analysis, although one obvious preliminary observation is that if voting concerns several referenda, as in fact it often does in practice, the importance of voting costs in selecting between high and low valuation individuals must be reduced. But the main difficulty with voting costs, and the reason we have abstracted from these costs in this paper, is their poor empirical record. Given the difficulty in explaining observed turn-outs, in large elections normative recommendations that rely on rational self-selection in the presence of voting costs seem particularly courageous.

If the test is finally empirical, then the bonus vote schemes should also be subjected to empirical validation, or more precisely, given that it does not exist, to experimental testing. It is this direction that we are pursuing currently.

Appendix A. Three preliminary lemmas

We begin by three preliminary results that will be used repeatedly. Define votes in favor as positive votes and votes against as negative votes, and the vote differential, the sum of all votes cast in referendum r, as V_r.

Lemma A.1. *Consider the voting problem in the absence of bonus votes when everybody votes sincerely. Call p_r a voter's probability of obtaining the desired outcome in referendum r. Then if $F_r(v)$ is symmetric around 0, $V_r \sim N(0,n)$ and $p_r \simeq 1/2 + 1/\sqrt{2\pi n}$.*

Proof of Lemma A.1. The derivation of the asymptotic distribution of the vote differential is standard (see for example Feller, 1968, pp. 179–182). The distribution is normal with mean given by the sample mean $(1/2(-1)+1/2(1)=0)$ and variance given by the sum of the variances of the summands: $n[(1/2)(-1)^2+(1/2)(1)^2]=n$. Because the distribution does not depend on $F_r(v)$, we can ignore the subscript r. Taking into account possible ties, the probability of obtaining one's desired outcome is:

$$p = \text{prob}(V_i \le 0) + (1/2)\text{prob}(V_i = 1),$$

where V_i is the voting differential excluding voter i. Given the discreteness of the votes:

$$\text{prob}(V_i \le 0) \simeq \frac{1}{2}\left(1 + \frac{1}{\sqrt{2\pi(n-1)}}e^{-0}\right) = \frac{1}{2}\left(1 + \frac{1}{\sqrt{2\pi(n-1)}}\right)$$
$$(1/2)\text{prob}(V_i = 1) \simeq \frac{1}{2}\left(\frac{1}{\sqrt{2\pi(n-1)}}e^{-\frac{(1)^2}{2(n-1)}}\right) \simeq \frac{1}{2}\left[\frac{1}{\sqrt{2\pi(n-1)}}\left(1-\frac{1}{2(n-1)}\right)\right] = \frac{1}{2}\left(\frac{1}{\sqrt{2\pi(n-1)}}-O\left(n^{-3/2}\right)\right).$$

Given n large and ignoring terms of order $O\left(n^{-3/2}\right), p \sim 1/2 + 1/\sqrt{2\pi n}.$ □

[28] According to Gerber (1999), narrow business interests have a comparative advantage in influencing popular votes through financial resources, while grass-root movements have a comparative advantage in gathering votes. If this logic is correct, bonus votes would both help to protect resource-less minorities against the power of the majority and reduce the power of money in direct democracy.

[29] In the legal literature, Cooter (2002) compares “median democracy” (direct democracy) to “bargain democracy” (legislatures) and argues for the practical superiority of the former, while admitting that the latter is “ideally” more efficient. Increasing the representation of intense preferences in a direct democracy is a step towards higher efficiency.

[30] See the classic analysis by Ledyard (1984), and more recently, among others, Campbell (1999), Krasa and Polborn (2005), Taylor and Yildirim (2005).

When we add to the problem a bonus vote of arbitrary value, we need to be more careful about the discreteness of the asymptotic distribution of the vote differential. The subtlety is in sizing correctly the steps of the distribution. We begin by presenting the local limit theorem relevant to our problem:

Gnedenko local limit theorem (Gnedenko and Kolmogorov, 1968, ch.9). *Consider a random variable z. We say that z is distributed according to a lattice distribution if all values that z can assume with positive probability can be expressed as $a+s_jh$ with $h>0$ and s_j integer $\forall j$ (where j indexes any point in the support that has positive probability). Select the representation $z_j=a+s_jh^0$ such that h^0 is the largest common divisor of all possible pairwise differences $z_j-z_{j'}$, and consider the normalized random variable $z'\equiv(z-a)/h^0$. Call $Z'=\sum_{i=1}^{n} z'_i$ the sum of independent, identically distributed random variables z′. If z′ has finite mean E(z′) and non-zero variance $\sigma^2_{z'}$, then:*

$$\text{prob}\{Z'=y\} \rightarrow \frac{1}{\sigma_{z'}\sqrt{2\pi n}}\,\text{Exp}\left[\frac{-(y-nE(z'))^2}{2n\sigma^2_{z'}}\right] \text{ as } n\rightarrow\infty$$

For our purposes, in each referendum r, each voter has the same feasible options: $\mathbf{X}_r=\{\pm(1+\theta),\pm1\}$ with the constraint that the bonus vote can be cast only once. If voters use symmetric strategies, realized strategies x_{ir} are iid for all i's; we can drop the subscript i, and use Gnedenko's local limit theorem to characterize the asymptotic distribution of the votes differential. If necessary, we need to impose the correct normalization and will indicate by $\mathbf{X}'_r$ the set of feasible strategies to which Gnedenko's theorem can be applied. There are two cases: if $\theta\geq1$, no normalization is required, and $\mathbf{X}'_r=\{\pm(1+\theta),\pm1\}$; if $\theta<1$, set $h^0=\theta$ and $a=0$, and thus $\mathbf{X}'_r=\{\pm(1+1/\theta),\pm1/\theta\}$. Call $\phi_{x'r}$ the probability that any voter casts x' votes in referendum r, again distinguishing between positive and negative votes, and V'_r the votes' differential, in terms of normalized votes: $V'_r=\sum_{i=1}^{n} x'_{ir}$. Then: $E_r(x')\equiv\mu_{r'}=\sum_{x'\in\mathbf{X}'_r}\phi_{x'r}x'$; $\sigma_{r'}{}^2=\sum_{x'\in\mathbf{X}'_r}\phi_{x'r}(x'-\mu_r)^2$, and:

$$\text{prob}\{V'=y\}\rightarrow\frac{1}{\sigma_{r'}\sqrt{2\pi n}}\,\text{Exp}\left[\frac{-(y-n\mu_{r'})^2}{2n\sigma^2_{r'}}\right] \text{ as } n\rightarrow\infty.$$

The important observation is that the normalization leaves unchanged the relative value of the bonus vote, and thus all pivot probabilities can be read directly from the limit theorem applied to the normalized strategies.

The following Lemma will be needed to prove Lemma A.3 and Lemma 1.

Lemma A.2. *There is no symmetric equilibrium where there exists a referendum r such that $\phi_r=1$.*

Proof of Lemma A.2. As in the text, call ϕ_r a voter's ex ante probability of casting the bonus vote on referendum r, and thus with large n, iid valuations, and symmetric strategies, the expected fraction of voters casting their bonus vote on r. Suppose $\phi_r=1$, and thus voter i expects all other voters to cast their bonus vote on referendum r. All referenda are then decided by simple majority voting, since all voters in any given referendum are casting votes of equal value. Consider voter i's gain from casting his bonus vote in some other referendum s, as opposed to r. Such a gain is smallest with $\theta<1$, when switching the bonus vote from referendum r has an expected cost of $(1/2)\,v_{ir}\,\text{prob}(V'_r=I_r(1+1/\theta))$ (the possible loss of a tie in r), and an expected gain of $(1/2)\,v_{is}\text{prob}(V'_s=I_s)$ (the possible move from a tie to a victory in referendum s), where:

$$I_r=\begin{cases}-1 \text{ if } \mathrm{v_{ir}}>0\\ \;\;1 \text{ if } \mathrm{v_{ir}}<0\end{cases} \quad\text{and}\quad I_s=\begin{cases}-1 \text{ if } \mathrm{v_{is}}>0\\ \;\;1 \text{ if } \mathrm{v_{is}}<0\end{cases}.$$

If i is in favor of a proposal, his vote is positive and it is pivotal if it counters a negative vote differential of the same magnitude; the signs are reversed if i is against the proposal. With all votes in any given referendum being of equal value, both probabilities reduce to the probability that i's preferred side is losing by a single voter, which with all distributions $\mathbf{F}$ symmetric must be equal in the two referenda. (Formally, if $\phi_r=1$, $\mathbf{X}'_r=\mathbf{X}'_s=\{-1, 1\}$, and with $\mathbf{F}$ symmetric $\mu_{r'}=\mu_{s'}=0$, and $\sigma_{r'}=\sigma_{s'}=1$). Thus the probability that i finds it profitable to deviate is the probability that there exists at least one s such that $v_{is}>v_{ir}$, or the probability that v_{ir} is not i's highest valuations. With iid draws, such a probability is $1-1/k>0$. Hence $\phi_{ir}<1$ and there cannot be a symmetric equilibrium with $\phi_r=1$. □

As in the text, call $\mathbf{v}_i$ the vector of voter i's valuations, and $\boldsymbol{\nu}_i$ the vector of i's intensities (absolute valuations). Then:

Lemma A.3. *In equilibrium, $|x_{ir}(\mathbf{v_i},\mathbf{F})|=x_{ir}(\boldsymbol{\nu}_i,\mathbf{F})$ $\forall i,\forall r$: for all voters and in all referenda the number of votes cast is independent of the signs of the voter's valuations.*

Proof of Lemma A.3. Consider two referenda, r and s. Voter i is choosing whether to cast the bonus vote in referendum r or in referendum s.

Call $p_{\theta r}$ the probability of obtaining the desired outcome when casting the bonus vote, p_r the probability of obtaining the desired outcome when casting the regular vote only, and V'_r the normalized votes' differential in referendum r, excluding voter i: $V'_r = \sum_{m=1, m \neq i}^{n} x'_{mr}$. Suppose first $\theta \geq 1$ and $\mathbf{X}'_r = \{\pm(1+\theta), \pm 1\}$. Voter i's optimal strategy is to cast the bonus vote on s if and only if $v_s p_{\theta s} + v_r p_r > v_r p_{\theta r} + v_s p_s$. Given Lemma A.2 the asymptotic distributions of V'_r and V'_s have the same support, and we can write this condition as:

$$\frac{v_s}{v_r} > \frac{p_{\theta_r} - p_r}{p_{\theta_s} - p_s} = \frac{1/2[\text{prob}(V'_r = I_r(1+\theta)) + \text{prob}(V'_r = I_r)] + \sum_{m=1}^{\theta-1} \text{prob}(V'_r = I_r(1+m))}{1/2[\text{prob}(V'_s = I_s(1+\theta)) + \text{prob}(V'_s = I_s)] + \sum_{m=1}^{\theta-1} \text{prob}(V'_s = I_s(1+m))}. \tag{A.1}$$

Using the limit theorem, Eq. (A.1) becomes:

$$\frac{v_s}{v_r} > \frac{\sigma_{s'}}{\sigma_{r'}} \left(\frac{\text{Exp}\left[-\frac{(I_r(1+\theta)-(n-1)\mu_{r'})^2}{2(n-1)\sigma_{r'}^2}\right] + \text{Exp}\left[-\frac{(I_r-(n-1)\mu_{r'})^2}{2(n-1)\sigma_{r'}^2}\right] + \sum_{m=1}^{\theta-1} \text{Exp}\left[-\frac{(I_r(1+m)-(n-1)\mu_{r'})^2}{2(n-1)\sigma_{r'}^2}\right]}{\text{Exp}\left[-\frac{(I_s(1+\theta)-(n-1)\mu_{s'})^2}{2(n-1)\sigma_{s'}^2}\right] + \text{Exp}\left[-\frac{(I_s-(n-1)\mu_{s'})^2}{2(n-1)\sigma_{s'}^2}\right] + \sum_{m=1}^{\theta-1} \text{Exp}\left[-\frac{(I_s(1+m)-(n-1)\mu_{s'})^2}{2(n-1)\sigma_{s'}^2}\right]} \right) = \frac{\sigma_{s'}}{\sigma_{r'}} O\left(e^{-(n-1)\left(\mu_{r'}^2/\sigma_{r'}^2 - \mu_{s'}^2/\sigma_{s'}^2\right)}\right).$$

There are several possibilities. If $\mu_{r'} = \mu_{s'} = 0$, the condition simplifies to $v_s / v_r > \sigma_{s'} / \sigma_{r'}$. (More generally if $({\mu_{r'}}^2 / {\sigma_{r'}}^2 - {\mu_{s'}}^2 / {\sigma_{s'}}^2)$ is of order smaller or equal to $1/n$, the condition becomes $v_s / v_r > a(\sigma_{s'} / \sigma_{r'})$ where a is some positive constant). If ${\mu_{r'}}^2 / {\sigma_{r'}}^2 - {\mu_{s'}}^2 / {\sigma_{s'}}^2$ is negative and of order larger than $1/n$, the condition is never satisfied, and the bonus vote should not be spent on referendum s. Finally, if $({\mu_{r'}}^2 / {\sigma_{r'}}^2 - {\mu_{s'}}^2 / {\sigma_{s'}}^2)$ is positive and of order larger than $1/n$, the condition is always satisfied, and the bonus vote should not be spent on referendum r. In none of the cases is the choice of strategy affected by the sign of i's valuations or the direction of i's votes.

Suppose now that $\theta < 1$, and $\mathbf{X}'_r = \mathbf{X}'_s = \{\pm(1+1/\theta), \pm 1/\theta\}$. Then voter i should cast the bonus vote on s if and only if:

$$\frac{v_s}{v_r} > \frac{p_{\theta_r} - p_r}{p_{\theta_s} - p_s} = \frac{1/2[\text{prob}(V'_r = I_r(1+1/\theta)) + \text{prob}(V'_r = I_r/\theta)]}{1/2[\text{prob}(V'_s = I_s(1+1/\theta)) + \text{prob}(V'_s = I_s/\theta)]}. \tag{A.2}$$

It is trivial to verify that the same conclusion holds here. Lemma A.2 is established. □

If voter i's optimal strategy does not depend on the sign of i's valuations, then the following corollary follows immediately:

Corollary to Lemma A.3. If the distributions $\mathbf{F}$ are symmetrical around 0, in all equilibria $\mu_{r'} = 0$ and $E(V'_r) = 0$ $\forall r$.

Appendix B. Proofs of results in the text

Proof of Lemma 1. The lemma follows immediately from Lemma A.3 and its corollary. With $E(V'_r) = n\mu_r = 0$ $\forall r$, Eqs. (A.1) and (A.2) yield:

$$\frac{p_{\theta_r} - p_r}{p_{\theta_s} - p_s} = \frac{\sigma_{s'}}{\sigma_{r'}}.$$

If $\theta \geq 1$ and $\mathbf{X}'_r = \{\pm(1+\theta), \pm 1\}$, then ${\sigma_{r'}}^2 = \phi_r(1+\theta)^2 + (1-\phi_r) = 1 + \phi_r(2\theta + \theta^2)$; if $\theta < 1$ and $\mathbf{X}'_r = \{\pm(1+1/\theta), \pm 1/\theta\}$, then ${\sigma_{r'}}^2 = \phi_r(1+1/\theta)^2 + (1-\phi_r)(1/\theta)^2 = \theta^{-2}[1 + \phi_r(2\theta + \theta^2)]$. In either case

$$\frac{p_{\theta_r} - p_r}{p_{\theta_s} - p_s} = \sqrt{\frac{1 + \phi_s\left(2\theta + \theta^2\right)}{1 + \phi_r\left(2\theta + \theta^2\right)}}.$$

As Lemma 1 states, it follows that voter i's optimal strategy is to cast the bonus vote in referendum s if and only if:

$$\frac{v_{is}}{v_{ir}} > \sqrt{\frac{1 + \phi_s\left(\theta^2 + 2\theta\right)}{1 + \phi_r\left(\theta^2 + 2\theta\right)}} \quad \forall r \neq s.$$

The probabilities $p_{\theta r}$ and p_r can be read directly from the limit theorem. Given Lemma A.3, we can suppose, with no loss of generality, that voter i is against proposal r. If $\theta \geq 1$, p_r=prob($V'_r \leq 0$)+(1/2)prob(V'_r=1), and $p_{\theta r}$=prob($V'_r \leq 0$)+$\sum_{m=0}^{\theta-1}$ prob(V'_r=1+m)+(1/2)prob(V'_r=1+θ) or:

$$p_r \rightarrow \left(\frac{1}{2}+\frac{1}{2}\frac{1}{\sigma_{r'}\sqrt{2\pi(n-1)}}\right)+\frac{1}{2}\left(\frac{1}{\sigma_{r'}\sqrt{2\pi(n-1)}}\mathrm{Exp}\left[\frac{-1}{2n\sigma^2_{r'}}\right]\right)=\frac{1}{2}+\frac{1}{\sqrt{2\pi(n-1)\left[1+\phi_r\left(\theta^2+2\theta\right)\right]}}+O\left(n^{-3/2}\right);$$

$$p_{\theta_r} \rightarrow \left(\frac{1}{2}+\frac{1}{2}\frac{1}{\sigma_{r'}\sqrt{2\pi(n-1)}}\right)+\sum_{m=0}^{\theta-1}\left(\frac{1}{\sigma_{r'}\sqrt{2\pi(n-1)}}\mathrm{Exp}\left[\frac{-(1+m)^2}{2(n-1)\sigma^2_{r'}}\right]\right)+\frac{1}{2}\left(\frac{1}{\sigma_{r'}\sqrt{2\pi(n-1)}}\mathrm{Exp}\left[\frac{-(1+\theta)^2}{2(n-1)\sigma^2_{r'}}\right]\right)$$
$$=\frac{1}{2}+\frac{1+\theta}{\sqrt{2\pi(n-1)\left[1+\phi_r\left(\theta^2+2\theta\right)\right]}}+O\left(n^{-3/2}\right)$$

If instead $\theta<1$, then p_r=prob($V'_r \leq 0$)+$\sum_{m=0}^{1/\theta-2}$ prob(V'_r=1+m)+(1/2)prob(V'_r=1/θ), and $p_{\theta r}$=prob($V'_r \leq 0$)+$\sum_{m=0}^{1/\theta-1}$ prob(V'_r=1+m)+(1/2)prob(V'_r=1+1/θ), or:

$$p_r \rightarrow \left(\frac{1}{2}+\frac{1}{2}\frac{1}{\sigma_{r'}\sqrt{2\pi(n-1)}}\right)+\sum_{m=0}^{1/\theta-2}\frac{1}{2}\left(\frac{1}{\sigma_{r'}\sqrt{2\pi(n-1)}}\mathrm{Exp}\left[\frac{-(1+m)^2}{2(n-1)\sigma^2_{r'}}\right]\right)+\frac{1}{2}\left(\frac{1}{\sigma_{r'}\sqrt{2\pi(n-1)}}\mathrm{Exp}\left[\frac{-(1/\theta)^2}{2(n-1)\sigma^2_{r'}}\right]\right)$$
$$=\frac{1}{2}+\frac{1}{\theta}\frac{\theta}{\sqrt{2\pi(n-1)\left[1+\phi_r\left(\theta^2+2\theta\right)\right]}}+O\left(n^{-3/2}\right)=\frac{1}{2}+\frac{1}{\sqrt{2\pi(n-1)\left[1+\phi_r\left(\theta^2+2\theta\right)\right]}}+O\left(n^{-3/2}\right);$$

$$p_{\theta_r} \rightarrow \left(\frac{1}{2}+\frac{1}{2}\frac{1}{\sigma_{r'}\sqrt{2\pi(n-1)}}\right)+\sum_{m=0}^{1/\theta-1}\frac{1}{2}\left(\frac{1}{\sigma_{r'}\sqrt{2\pi(n-1)}}\mathrm{Exp}\left[\frac{-(1+m)^2}{2(n-1)\sigma^2_{r'}}\right]\right)+\frac{1}{2}\left(\frac{1}{\sigma_{r'}\sqrt{2\pi(n-1)}}\mathrm{Exp}\left[\frac{-(1+1/\theta)^2}{2(n-1)\sigma^2_{r'}}\right].\right)$$
$$=\frac{1}{2}+\left(\frac{1}{\theta}+1\right)\frac{\theta}{\sqrt{2\pi(n-1)\left[1+\phi_r\left(\theta^2+2\theta\right)\right]}}+O\left(n^{-3/2}\right)=\frac{1}{2}+\frac{1+\theta}{\sqrt{2\pi(n-1)\left[1+\phi_r\left(\theta^2+2\theta\right)\right]}}+O\left(n^{-3/2}\right)$$

With n large, these are the expressions in the Lemma. □

Proof of Proposition 1. Given Lemma 1, the proof of Proposition 1 is straightforward. (i) To see that the candidate strategy is indeed an equilibrium strategy, suppose all voters but i cast their bonus vote in the referendum with highest intensity. Since all intensities are drawn from the same probability distribution, with k draws each has probability $1/k$ of being the highest, implying ϕ_s=ϕ_r=1 / k ∀r. Thus the square root in Eq. (2) in the text equals 1 and by Lemma 1 voter i should follow the same strategy, establishing that it is indeed an equilibrium. (ii) To see that the equilibrium is unique, suppose to the contrary that there is an equilibrium where not all ϕ_r, 's are equal, and call s the referendum such that ϕ_s=max{ϕ_r}. Then $\sqrt{\left[1+\phi_s\left(\theta^2+2\theta\right)\right]/\left[1+\phi_r\left(\theta^2+2\theta\right)\right]} \equiv \alpha_{sr} \geq 1$ ∀r, with at least one strict inequality. Call r' one of the referenda for which the strict inequality holds (at least one r' must exist). Then, by Eq. (2), in the bilateral comparison between s and r' the expected share of voters casting their bonus vote on s is lower than on r'. We can have ϕ_s=max{ϕ_r} only if there exists at least one r'' such that $\alpha_{sr''}<\alpha_{r'r''}$. But this requires $\phi_{r'}>\phi_s$, contradicting ϕ_s=max{ϕ_r}. □

Proof of Proposition 3. At θ=0, ω=1—as must be the case by the definition of ω and as can be verified by setting θ=0 in Eq. (13). Proposition 3 must hold if ω is increasing in θ at θ=0: although θ must be a rational number larger than $1/\sqrt{n}$, there is always a value of n such that θ can take values arbitrarily close to 0. Because in addition ω is continuous in θ in the neighborhood of θ=0, in this neighborhood we can treat θ as a continuous variable. Taking into account Eqs. (11) and (12), the derivative dω/dθ is greatly simplified by being evaluated at θ=0. In fact it is not difficult to show that it reduces to:

$$\left.\frac{\mathrm{d}\omega}{\mathrm{d}\theta}\right|_{\theta=0}=\sum_{r=1}^{k}\left[\left(\int_0^1\prod_{s\neq r}G_s(v)vg_r(v)\mathrm{d}v\right)\right]-\sum_{r-1}^{k}\left[Ev_r(v)\left(\int_0^1\prod_{s\neq r}G_s(v)g_r(v)dv\right)\right]. \tag{A3}$$

Integrating by part the first summation, we obtain:

$$\sum_{r=1}^{k}\left[\left(\int_0^1\prod_{s\neq r}G_s(v)vg_r(v)\mathrm{d}v\right)\right]=\int_0^1\left(1-\prod_{r=1}^{k}G_r(v)\right)\mathrm{d}v=Ev_{(k)},$$

where $Ev_{(k)}$ now stands for the expected highest intensity over all distributions. Thus Eq. (A3) can be rewritten more simply as:

$$\left.\frac{\mathrm{d}\omega}{\mathrm{d}\theta}\right|_{\theta=0}=Ev_{(k)}-\sum_{r=1}^{k}Ev_r\,\phi_r|_{\theta=0}.$$

But since $\phi_r|_{\theta=0} \in (0,1)$ ∀$_r$ and $\sum_{r=1}^{k}\phi_r|_{\theta=0}=1$, the expression must be strictly positive, and the proposition is established. □

Proof of Proposition 3b. The proof proceeds identically to the proof of Proposition 3. Indeed, $\frac{d\omega}{d\theta}\Big|_{\theta=0} = \frac{d\omega^R}{d\theta}\Big|_{\theta=0}$ where $\omega^R \equiv \omega(\alpha_{sr}=1\forall s, r)$, Eq. (A3) continues to hold, and the argument is unchanged. □

Proof of Example 2. Here it turns out to be easier to work with $\tau \equiv 1/\theta$, the value of the regular votes relative to the bonus vote. With power distributions, the condition $\omega > 1$ then corresponds to:

$$\omega > 1 \Leftrightarrow \sum_{r=1}^{k} \left(\frac{b_r c_r}{\sum_{r=1}^{k} b_r + 1} \right) + \tau \sum_{r=1}^{k} \left(\frac{b_r c_r}{1+b_r} \right) > \sum_{r=1}^{k} \left(\frac{b_r}{1+b_r} \right) \tag{A4}$$

where:

$$c_r \equiv \sqrt{\frac{\sum_{r=1}^{k} b_r}{b_r(1+2\tau)+\tau^2}}.$$

The proof proceeds in three steps. First, we know from the proof of Proposition 3 that as τ approaches infinity, or equivalently θ approaches 0, ω approaches 1 from above. This immediately establishes that either $\omega > 1 \forall \tau$, or there exists at least one internal maximum at a finite value of τ. Second, we can derive the first-order condition that an internal maximum, if it exists, must satisfy. Differentiating the left hand side of Eq. (A4) with respect to τ, we find that the first derivative equals zero at some τ^* defined by the implicit equation:

$$\tau^* = \sum_{r=1}^{k} \left(\frac{\gamma_r(\tau^*)}{\sum_{r=1}^{k} \gamma_r(\tau^*)} \right) b_r, \tag{A5}$$

where

$$\gamma_r(\tau^*) \equiv \frac{b_r}{1+b_r} \left[\sum_{s \neq r} b_s \left(\frac{1}{\sum_{j=1}^{k} b_j \tau^{*2} + b_r(1+2\tau^*)} \right)^{3/2} \right].$$

For our purposes, the important point is that any and all τ^* must be a weighted average of the distribution parameters $\{b_1,\ldots, b_k\}$, with weights $\omega_r(\tau^*) \equiv \left(\gamma_r(\tau^*)/\sum_{r=1}^{k} \gamma_r(\tau^*)\right) \forall r = 1, \ldots, k$ and such that $\sum_{r=1}^{k} \omega_r(\tau^*) = 1$. In particular, each weight is strictly between 0 and 1 for all positive finite b_r and τ^* (including in the limit as τ^* approaches 0). Thus any and all τ^* must satisfy $\tau^* < b_k$, where $b_k \equiv \max\{b_r\}$. Third, consider the limit of ω as τ approaches 0:

$$\lim_{\tau->0} \omega = \left(\frac{\sqrt{\sum_{r=1}^{k} b_r}\left(\sum_{r=1}^{k} \sqrt{b_r}\right)}{1+\sum_{r=1}^{k} b_r} \right) / \left(\sum_{r=1}^{k} \left(\frac{b_r}{1+b_r} \right) \right). \tag{A6}$$

The limit is positive and finite. There are two possibilities. If Eq. (A6) is smaller than 1, then by step 1 above an internal maximum must exist. Call $\tau^{*\prime}$ the largest value of τ^* that satisfies Eq. (A5), and $\omega(\tau^{*\prime})$ must be a maximum: then $\omega > 1 \forall \tau > \tau^{*\prime}$. And since $\tau^{*\prime} < b_k$, $\omega > 1$ $\forall \tau \geq b_k$. If Eq. (A6) is larger than 1, either no internal maximum exists and ω is larger than 1 for all τ—in which case, $\omega > 1$ $\forall \tau \geq b_k$ is trivially satisfied. Or an internal maximum exists, and the argument above continues to hold: $\omega > 1 \forall \tau > \tau^{*\prime}$, and since $\tau^{*\prime} < b_k$, $\omega > 1$ $\forall \tau \geq b_k$. Thus in all cases, $\omega > 1$ $\forall \tau \geq b_k$ or, equivalently, $\omega > 1$ $\forall \theta \leq 1/b_k$. □

Proof of Proposition 4. Call p_{sr} the probability of obtaining one's desired outcome in referendum r when the probability of approval ψ_r is stochastic and the voter does not cast the bonus vote, and $p_{s\theta r}$ the corresponding probability when the voter does cast the bonus vote in referendum r. The notation will be simplified by writing $\psi_r \equiv 1/2 + \delta_r$ where δ_r is distributed according to H_δ defined over the support $[-1/2, 1/2]$ and symmetric around 0, and where each realized δ_r is an independent draw from H_δ. Given the equilibrium strategy and $G_r(v) = G(v)$ for all r, it follows that $p_{sr} = p_s$ and $p_{s\theta r} = p_{s\theta}$ for all r.

For given δ_r, the vote differential in referendum r continues to be distributed according to a Normal distribution with mean given by the sample mean, and variance given by the sample variance. Given the equilibrium strategy, $EV(\delta_r) = n[\psi_r(1/k)(1+\theta) + \psi_r(1-1/k)(1) + (1-\psi_r)(1/k)(-1-\theta) + (1-\psi_r)(1-1/k)(-1)] = n[(2\psi_r - 1)(1+\theta/k)]$, where $\psi_r = \text{prob}(v_{ir} > 0)$ for all i,r and $1/k = \text{prob}(v_{ir} = \max\{v_i\})$ over all r, and thus $1/k = \text{prob}(|x_{ir}| = 1+\theta)$. Using $\psi_r = 1/2 + \delta_r$, we can write $EV(\delta_r) = 2n\delta_r(1+\theta/k)$. Similarly, the variance of the votes differential $\sigma_r^2(\delta_r)$ is given by $n[(\psi_r + 1 - \psi_r)(1/k)(1+\theta)^2 + (\psi_r + 1 - \psi_r)(1-1/k)(1) + O(\delta_r^2)] = n[(\theta^2 + 2\theta + k)/k + O(\delta_r^2)]$. Taking into account the discreteness of the distribution and ignoring terms of order δ_r^2 and higher, p_s and $p_{\theta s}$ can be approximated by:

$$\begin{aligned} p_s(\delta) &\simeq \widetilde{\Phi}(0) + \frac{\sqrt{k}}{\sqrt{2\pi n\left(k+\theta^2+2\theta\right)}} \operatorname{Exp}\left[-\frac{n 4 \delta^2 k (1+\theta/k)^2}{2\left(k+\theta^2+2\theta\right)} \right]; \\ p_{\theta_s}(\delta) &\simeq \widetilde{\Phi}(0) + \frac{(1+\theta)\sqrt{k}}{\sqrt{2\pi n\left(k+\theta^2+2\theta\right)}} \operatorname{Exp}\left[-\frac{n 4 \delta^2 k (1+\theta/k)^2}{2\left(k+\theta^2+2\theta\right)} \right]. \end{aligned} \tag{A7}$$

where $\tilde{\Phi}(0)$ is the cumulative function at 0 of a Normal distribution with mean $2\delta n(1+\theta/k)$ and variance $n(k+\theta^2+2\theta)/k$. (The probabilities simplify to the values in Eq. (1) in the text for $\delta=0$).

Taking into account that the expected intensity in each referendum is independent of the direction of the preferences, we can write ex ante expected utility with stochastic approval, EU_s, as:

$$EU_s = \int_{-1/2}^{1/2} \left[k(Ev)p_s(\delta) + Ev_{(k)}(p_{s\theta}(\delta) - p_s(\delta))\right] dH_\delta(\delta) \tag{A8}$$

With H_ψ symmetric around 1/2, $\int_{-1/2}^{1/2} \tilde{\Phi}(0) dH_\delta(\delta) = 1/2$. Thus:

$$EU_s - ER = \left[k(Ev) + \theta Ev_{(k)}\right] \int_{-1/2}^{1/2} \frac{\sqrt{k}}{\sqrt{2\pi n\left(k+\theta^2+2\theta\right)}} \operatorname{Exp}\left[-\frac{n4\delta^2 k(1+\theta/k)^2}{2\left(k+\theta^2+2\theta\right)}\right] dH_\delta(\delta).$$

At large n, only realizations of δ close to 0 yield positive probabilities. The integral term can be solved as:[31]

$$\int_{-1/2}^{1/2} \frac{\sqrt{k}}{\sqrt{2\pi n\left(k+\theta^2+2\theta\right)}} \exp\left[-\frac{n4\delta^2 k(1+\theta/k)^2}{2\left(k+\theta^2+2\theta\right)}\right] dH_\delta(\delta) = \frac{\sqrt{k}}{\sqrt{2\pi n\left(k+\theta^2+2\theta\right)}} \sqrt{2\pi} \frac{\sqrt{\left(k+\theta^2+2\theta\right)}}{\sqrt{kn}2(1+\theta/k)} h_\delta(0)$$
$$= \frac{1}{2(1+\theta/k)} \frac{1}{n} h_\delta(0).$$

Therefore:

$$EU_s - ER = \left[k(Ev) + \theta Ev_{(k)}\right] \left[\frac{1}{2(1+\theta/k)} \frac{1}{n} h_\delta(0)\right]. \tag{A9}$$

At $\theta=0$, Eq. (A9) reduces to the expected improvement over randomness with simple majority, or:

$$EW_s - ER = k(Ev) \left[\frac{1}{2}\frac{1}{n} h_\delta(0)\right].$$

We obtain:

$$\omega_s = \frac{k(Ev) + \theta Ev_{(k)}}{(Ev)(k+\theta)}$$

It is then immediate that $\omega_s > 1 \Leftrightarrow Ev_{(k)} > Ev$, a condition that is always satisfied. □

Proof of Proposition 5. The proof proceeds by showing that if $P(v)$ first-order stochastically dominates $C(v)$ for all r, each referendum passes with probability approaching 1. With a large population this outcome is ex ante efficient and dominates the outcome of simple majority voting. Recall:

$$\phi_P = \left(\frac{1}{2}\right)^{k-1} \left[\sum_{s=0}^{k-1} \left(\binom{k-1}{s} \int_0^1 C(v)^{k-1-s} P(v)^s p(v) dv\right.\right];$$

$$\phi_C = \left(\frac{1}{2}\right)^{k-1} \left[\sum_{s=0}^{k-1} \left(\binom{k-1}{s} \int_0^1 P(v)^{k-1-s} C(v)^s c(v) dv\right.\right];$$

[31] Exploiting the Gaussian integral:

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\pi/a}$$

with large n:

$$\int_{-c}^{c} g(x) e^{-\frac{nx^2}{2q}} dx = g(0)\sqrt{2q\pi/n}.$$

or identically, using the index $j \equiv k-1-s$:

$$\phi_C = \left(\frac{1}{2}\right)^{k-1}\left[\sum_{j=0}^{k-1}\binom{k-1}{j}\int_0^1 C(v)^{k-1-j}P(v)^j c(v)\mathrm{d}v\right].$$

Because both $P(v)$ and $C(v)$ are strictly increasing in v, and $P(v)$ first- order stochastically dominates $C(v)$, each term summed in ϕ_P is larger than its corresponding term in ϕ_C, and thus $\phi_P > \phi_C$. The vote differential in each referendum is normally distributed with mean $\mathrm{EV}=(n/2)\theta(\phi_P-\phi_C)>0$ and variance $\sigma_V{}^2=(n/2)[(\phi_P+\phi_C)(2\theta+\theta^2)-\theta^2/2(\phi_P-\phi_C)^2+2]$. Recall that $\Phi(x)\simeq 1-x^{-1}\frac{1}{\sqrt{2\pi}}\,Exp\left[\frac{-x^2}{2}\right]$ when x is large and $\Phi(\cdot)$ is the standard Normal distribution function (see for example Feller, 1968, chapter 7). Hence:

$$\mathrm{prob}(V>0)=\mathrm{prob}\left[\left(\frac{V-EV}{\sigma_V}\right)>\frac{\mathrm{EV}}{\sigma_V}\right]=\Phi\left(\frac{\mathrm{EV}}{\sigma_V}\right)\simeq 1-\frac{1}{\sqrt{2\pi}}\frac{\sigma_V}{EV}e^{-EV^2/\left(2\sigma_V^2\right)}=1-\frac{1}{\sqrt{2\pi n}}O(e^{-n}),$$

and the probability that proposal r passes equals

$$\mathrm{prob}(V>0)+\frac{1}{2}\mathrm{prob}(V=0)\simeq 1-\frac{1}{2}\frac{1}{\sqrt{2\pi n}}O(e^{-n})\simeq 1.$$

Thus a proposal passes with probability approaching 1.

We can then write ex ante utility as:

$$\mathrm{EU}\simeq\sum_r\frac{1}{2}E_{Pr}(v),$$

where 1/2 is the ex ante probability of being in favor of any proposal (given the 0 median). With simple majority voting, on the other hand:

$$\mathrm{EW}\simeq\sum_r\left(\frac{1}{2}+\frac{1}{\sqrt{2\pi n}}\right)\left(\frac{E_{Pr}(v)+E_{Cr}(v)}{2}\right). \tag{A10}$$

Because $E_{Pr}\,(v)>E_{cr}\,(v)$, we can conclude that there always exists a large but finite $\tilde{n}$ such that for all $n>\tilde{n}$, EU>EW. The result holds for any positive θ, independently of its precise value. In addition, if we consider a sequence of referenda with increasing n, as $n\to\infty$, $\mathrm{EW}_n\to\mathrm{ER}=1/2\sum_r\left(\frac{E_{Pr}(v)+E_{Cr}(v)}{2}\right)$, while $\mathrm{EU}_n\to\sum_r\frac{1}{2}E_{\mathrm{P}r}(v)$ yielding the Corollary to Proposition 5 in the text. □

Proof of Examples 4 and 5. It is useful to begin with the logic of the proof. In each referendum, the asymptotic distribution of the votes differential V_r is a discrete distribution with steps at all points in the support corresponding to values that V_r can assume with positive probability. In our problem, the difficulty is that the support of the distribution depends on the voters' strategies and thus is determined in equilibrium. Call the value of each bonus vote ϑ, and conjecture an equilibrium where all voters cumulate their bonus vote on their highest intensity. If in such an equilibrium the minimum increments in the possible values of V_r are identical across referenda and equal $\min(1,\vartheta)$, (thus equal to the minimum change in the number of votes available to the players), then for n large, the scenario is indeed an equilibrium. If in addition the condition on the minimum increments in the possible values of V_r is satisfied for all possible voters' strategies, then the equilibrium where all bonus votes are cumulated on each voter's highest valuation is unique.

1. Consider first example 4, where $\vartheta=1$.

(i) To verify that cumulating all bonus votes is an equilibrium, consider voter i's choices when every other voter follows this strategy. In each referendum all voters $j\neq i$ choose from the set of strategies: $\{-(1+k-1),-1,1,(1+k-1)\}$. Given iid valuation draws, the distribution of the expected votes' differential is identical across referenda, and we drop the index r. As argued earlier, i's optimal strategy cannot depend on the sign of his valuations, and with no loss of generality let us suppose that i favors all referenda. If i cumulates all bonus votes on his highest intensity, i's expected utility is:

$$\mathrm{EU}_{ci}=v_{i(k)}[\mathrm{prob}(V>-k)+1/2\mathrm{prob}(V=-k)]+\sum\nolimits_{m=1}^{k-1}v_{i(m)}[\mathrm{prob}(V>-1)+1/2\mathrm{prob}(V=-1)],$$

where $v_{i(m)}$ is i's mth highest intensity (thus $v_{i(k)}$ is the highest and $v_{i(1)}$ the lowest), and the index c in EU_{ci} stands for "cumulating".

Suppose i were to deviate and cast $(k-1-y)$ bonus votes on the referendum with the highest intensity and $y\leq k-1$ on the referendum with the second highest. Then i's expected utility would be:

$$\begin{aligned}\mathrm{EU}_{di}=\;&v_{i(k)}[\mathrm{prob}(V>-(k-y))+1/2\mathrm{prob}(V=-(k-y))]+v_{i(k-1)}[\mathrm{prob}(V>-(1+y))+1/2\mathrm{prob}(V=-(1+y))]\\&+\sum_{m=1}^{k-2}v_{i(m)}[\mathrm{prob}(V>-1)+1/2\mathrm{prob}(V=-1)],\end{aligned}$$

where the index d stands for "deviation". The difference in expected utility then is:

$$\mathrm{EU}_{di}-EU_{ci} = v_{i(k-1)}\left[1/2prob(V=-1)+1/2prob(V=-(1+y))+\sum_{l=1}^{y-1}prob(V=-(1+l))\right]$$
$$-v_{i(k)}\left[1/2prob(V=-(k-y))+1/2prob(V=-k)+\sum_{l=1}^{y-1}prob(V=-(k-l))\right].$$

Or:

$$\mathrm{EU}_{di}<\mathrm{EU}_{ci}\Leftrightarrow\frac{v_{i(k-1)}}{v_{i(k)}}<\frac{1/2\mathrm{prob}(V=-(k-y))+1/2\mathrm{prob}(V=-k)+\sum_{l=1}^{y-1}\mathrm{prob}(V=-(k-l))}{1/2\mathrm{prob}(V=-1)+1/2\mathrm{prob}(V=-(1+y))+\sum_{l=1}^{y-1}\mathrm{prob}(V=-(1+l))}$$
$$=\frac{1/2\mathrm{Exp}\left[\frac{-(k-y)^2}{2n\sigma^2}\right]+1/2\mathrm{Exp}\left[\frac{-k^2}{2n\sigma^2}\right]+\sum_{l=1}^{y-1}\mathrm{Exp}\left[\frac{-(k-l)^2}{2n\sigma^2}\right]}{1/2\mathrm{Exp}\left[\frac{-l^2}{2n\sigma^2}\right]+1/2\mathrm{Exp}\left[\frac{-(1+y)^2}{2n\sigma^2}\right]+\sum_{l=1}^{y-1}\mathrm{Exp}\left[\frac{-(1+l)^2}{2n\sigma^2}\right]}\to 1 \text{ for } n \text{ large}.$$

But $v_{i(k-1)}<v_{i(k)}$ and thus the deviation is not profitable. Could other deviations be profitable? It is not difficult to verify that any deviation that shifts bonus votes towards still lower valuations is strictly less profitable than the deviation analyzed above. Take *any* strategy where i casts y_m bonus votes on the referendum with intensity $v_{i(m)}$ and $y_{m'}>0$ on referendum with intensity $v_{i(m')}>v_{i(m)}$. If i shifts the $y_{m'}$ bonus votes towards the referendum with intensity $v_{i(m)}$, the change in expected utility is:

$$\Delta\mathrm{EU}_{m+1}^{m} = v_{i(m)}\left[1/2\mathrm{prob}(V=-(1+y_m))+1/2\mathrm{prob}(V=-(1+y_m+y_{m'}))+\sum_{l=1}^{y_{m'}-1}\mathrm{prob}(V=-(1+y_m+l))\right]$$
$$-v_{i(m')}\left[1/2prob(V=-(1+y_{m'}))+1/2prob(V=-1)+\sum_{l=1}^{y_{m'}-1}prob(V=-(1+y_{m'}+l))\right],$$

or, applying the limit theorem, $\Delta\mathrm{EU}_{m'}^{m}<0$ for all n large, because as above the two terms in brackets converge to the same value. Thus the deviation considered earlier is the most profitable, and cumulating all bonus votes on one's highest intensity is an equilibrium.

(ii) The result $\Delta\mathrm{EU}_{m'}^{m}<0$ holds true for any valuation-responsive strategy followed by all other voters, because with $k-1$ bonus votes, any such strategy implies that in all referenda some of the voters are expected to cast only their regular vote. Thus the correct step size in the asymptotic distribution of the vote differential is unity, and the expression for $\Delta\mathrm{EU}_{m'}^{m}$ applies unchanged. It follows then that cumulating all bonus votes on one highest intensity must be the unique equilibrium in responsive strategies. Finally, notice that the total number of bonus votes played no part in establishing (i); the equilibrium continues to hold for any number of bonus votes (although we have not shown that it remains unique).

2. Consider now Example 5, where $\vartheta=1/k$. In each referendum, the number of votes each voter $j\neq i$ casts belongs to the set: $\{-(1+(k-1)/k), -1, 1, (1+(k-1)/k)\}$, or, normalizing: $\{-(2k-1), -k, k, (2k-1)\}$. The normalization expresses all strategies in terms of the smallest shift in votes available to voter i and safeguards against mistakes. The normalized support of the asymptotic distribution of the votes' differential has steps of unit size, and the proof of part (i) of Example 4 applies here, with the appropriate changes in the number of votes cast. If i cumulates all bonus votes on his highest intensity, i's expected utility is:

$$\mathrm{EU}_{ci} = v_{i(k)}[\mathrm{prob}(V>-(2k-1))+1/2\mathrm{prob}(V=-(2k-1))]+\sum_{m=1}^{k-1}v_{i(m)}[\mathrm{prob}(V>-k)+1/2\mathrm{prob}(V=-k)].$$

Suppose i were to deviate and cast $(2k-1-y)$ votes on the referendum with highest intensity, and $(y+k)$ votes on the second highest. Voter i's expected utility would be:

$$\mathrm{EU}_{di} = v_{i(k)}[\mathrm{prob}(V>-(2k-1-y))+1/2\mathrm{prob}(V=-(2k-1-y))]+v_{i(k-1))}[\mathrm{prob}(V>-(k+y))+1/2\mathrm{prob}(V=-(k+y))]$$
$$+\sum_{m=1}^{k-2}v_{i(m)}[\mathrm{prob}(V>-k)+1/2\mathrm{prob}(V=-k)].$$

where again the index d stands for "deviation". The difference in expected utility then is:

$$\mathrm{EU}_{d2i}-EU_{ci} = v_{i(k-1)}\left[1/2prob(V=-k)+1/2prob(V=-(k+y))+\sum_{l=1}^{y-1}prob(V=-(k+l))\right]$$
$$-v_{i(k)}\left[1/2prob(V=-(2k-1-y))+1/2prob(V=-(2k-1))+\sum_{l=1}^{y-1}prob(V=-(2k-1-l))\right],$$

which must be negative for n large. The identical reasoning used earlier establishes that any deviation that shifts bonus votes towards referenda with still lower intensities must be strictly less profitable, establishing that cumulating all bonus votes on the highest intensity is indeed $\hat{\imath}$'s best response, and thus an equilibrium. Note that this last step in the proof follows if all other voters cumulate their bonus votes on the highest intensity, and thus $\mathbf{X}_r = \{-(2_k-1), -k, k, (2k-1)\}\ \forall_r$, or more precisely if the normalized support of the asymptotic distribution of the votes' differential has steps of unit size, in terms of the smallest change in the number of voters available to voter i. Such a condition guarantees that all terms in $\Delta \mathrm{EU}^m_{m'}$, are different from zero, and the terms in brackets converge to the same value. But with $\vartheta = 1/k$, the condition need not be satisfied for all possible strategies employed by the other voters, and equilibrium uniqueness cannot be claimed.

Finally, suppose voters were granted k votes of value $\vartheta = 1/k$. We show here that always cumulating all bonus vote on one' highest intensity is not an equilibrium. Suppose all other voters $j \neq 1$ cumulate their bonus votes, and thus in each referendum the number of votes each of them cast belongs to the set $\{-2, -1, 1, 2\}$. Notice that now the increments in the support of the asymptotic distribution of the votes differential V are larger than $\vartheta < 1$ (or, equivalently, the normalized strategies in each referendum belong to the set $\{-2k, -k, k, 2k\}$, implying that, in terms of the smallest feasible shift in the number of votes, the increments in the normalized support of the votes differential have size $k > 1$). If i cumulates all bonus votes on his highest intensity, i's expected utility is:

$$\mathrm{EU}_{ci} = v_{i(k)}[\mathrm{prob}(V > -2k) + 1/2\mathrm{prob}(V = -2k)] + \sum_{m=1}^{k-1} v_{i(m)}[\mathrm{prob}(V > -k) + 1/2\mathrm{prob}(V = -k)].$$

Consider a deviation where voter i were to cast a single bonus vote on all referenda. Then:

$$\mathrm{EU}_{di} = \sum_{m=1}^{k} v_{i(m)}[\mathrm{prob}(V > -(k+1)) + 1/2\mathrm{prob}(V = -(k+1))],$$

where, taking into account the possible values of the votes differential when all other voters cumulate their bonus votes:

$$\begin{aligned}
&\mathrm{prob}(V > -2k) = \mathrm{prob}(V = -k) + \mathrm{prob}(V > -k);\\
&\mathrm{prob}(V > -(k+1)) = \mathrm{prob}(V = -k) + \mathrm{prob}(V > -k);\\
&\mathrm{prob}(V = -(k+1)) = 0.
\end{aligned}$$

Thus i would gain from deviating if:

$$\sum_{m=1}^{k-1} v_{i(m)}[1/2\mathrm{prob}(V = -k)] > v_{i(k)}[1/2\mathrm{prob}(V = -2k)$$

or:

$$\frac{\sum_{m=1}^{k-1} v_{i(m)}}{v_{i(k)}} > \frac{1/2\,\mathrm{Exp}\left[\frac{-k^2}{2n\sigma^2}\right]}{1/2\,\mathrm{Exp}\left[\frac{-4k^2}{2n\sigma^2}\right]} \rightarrow 1 \text{ for } n \text{ large}.$$

Whenever $\sum_{m=1}^{k-1} v_{i(m)} > v_{(k)}$, an event that always has positive probability, the deviation is profitable. Hence with n large, all voters always cumulating their bonus votes on the referendum with highest intensity is not an equilibrium. □

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