# What Price Index Should Central Banks Target? An Open Economy Analysis*

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**Abstract**

There is currently a debate about what price index central banks should target when economies are open and exposed to international price shocks. This paper derives the optimal price index by solving the Ramsey problem in a New Keynesian small open economy model with an arbitrary number of sectors. This approach improves on existing theoretical benchmarks because (1) it makes an explicit distinction between the consumer price index (CPI) and the producer price index (PPI), and (2) it allows exogenous international price shocks to play a role. Qualitatively, I use the analytical expression of the optimal price index to discuss that popular indices, such as the PPI and the core/headline CPI, are suboptimal because they ignore the heterogeneity in price stickiness and the effect of inflation on the trade surplus. Quantitatively, I calibrate a 35-sector version of the model for 40 countries and show that stabilizing the optimal price index yields significantly higher welfare than alternative indices.

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# 1 Introduction

As many small open economies (SOEs) have shifted their monetary policy from exchange rate pegs to inflation targeting policies, there has been growing interest in which price index they should target. The theory of optimal monetary policy with a multi-sector economy can be used to answer this question, as in Aoki [2001] and Woodford [2010], but such analyses so far have been limited to closed economy setups, leaving open economy questions unanswered, such as the effect of international commodity prices and the role of trade patterns. This lack of the optimal price index theory in an open economy underlies the ongoing debate over the choice between, for instance, the headline consumer price index (CPI) versus the core CPI or the CPI versus the producer price index (PPI).

In this paper, I derive the optimal price index for open economies to stabilize by solving the problem of a central bank attempting to maximize household welfare, i.e., a Ramsey problem. I call the derived index the Ramsey price index (RPI) and present its analytical formula. Due to the openness of my model, the index depends on the export share of output in each sector in addition to the parameters studied in closed economy models such as the consumption share, price stickiness and the elasticity of substitution. By calibrating the model to 40 countries with 35 sectors, I find that (1) RPI stabilization performs better for all countries in terms of welfare than headline CPI, core CPI, or PPI stabilization and (2) the ranking of the indices other than the RPI differs across countries.

To derive the optimal price index, I begin with the multi-sector DSGE model with output price stickiness analyzed in Woodford [2010]. The use of a multi-sector model is necessary to answer my research question since different price indices arise due to the difference in weights applied to the prices in different sectors. Output price stickiness is the key monetary friction in my model and the workhorse model in the literature, in keeping with extensive empirical evidence (see Nakamura and Steinsson 2008, for example). Under output price stickiness, volatile inflation causes mispricing by firms, leading to welfare-damaging inefficient production activities.

As the key departure from Woodford [2010], I allow each sector in the economy to export a part of its output. This openness allows for a difference between CPI and PPI because when the economy can trade, what is produced is not necessarily consumed. The choice between the two indices is often the focus of monetary policy discussions especially for commodity exporters and developing countries. For instance, Frankel [2010] numerically analyzes Latin American commodity exporters and concludes that producer price based indices better perform than consumer price based indices in terms of price stability. India changed its target index from PPI to CPI in 2016; see Rajan [2016]. The existing theoretical framework is not suitable to answer this type of question since the consumption based weight coincides with the production based weight[1].

Another key feature of my model is the use of an SOE setup rather than a two-country setup. This is to capture the notion of international price movement that is exogenous to the economy. The Bank of Japan, for example, argued that the movement of the international oil price was the most important reason that it failed to achieve its inflation target; see

[1] The input-output structure is another reason that the PPI and CPI can differ. I focus on the difference arising from the trade in this paper.

Kawamoto and Nakahama [2017]. The SOE framework allows me to answer the question of whether the economy should bear such volatility in inflation that is caused by international price changes.

In this multi-sector New Keynesian (NK) SOE DSGE environment, I solve the Ramsey problem and obtain the optimal price index that remains constant in the long-run expectation under the Ramsey solution. This means that my proposed optimal price index is based on welfare maximization rather than an arbitrary objective. The welfare maximization problem is subject to optimizing behaviors of the representative household and firms under monetary frictions. The use of the Ramsey framework also means that the monetary policy considered in this paper is not limited to a particular class of monetary policy such as the Taylor rule. Despite the generality of the choice of monetary policy, I show that, in the long-run, a particular price index remains constant. I explore the property of this RPI qualitatively and quantitatively.

The key trade-off between stabilizing one price index versus another can be understood by considering the cost of volatile inflation rates in the sectors with lower weight in each price index. Therefore, the resulting optimal price index takes the form of a weighted sum of the prices in different sectors, where the weight assigned to each sector reflects the cost of inflation in each sector. In other words, in a multi-sector environment, the inflation rates of all the sectors cannot be stabilized simultaneously following a shock that leads to a relative price change. For example, when a change in world demand lowers the efficient relative price of oil, the central bank needs to essentially choose one of two options: (1) a stable oil price and an increase in non-oil price and (2) a stable non-oil price and a decrease in the oil price. Given this trade-off, we should stabilize the price of the sector with the higher cost of inflation.

My first main result is the analytical formula for the RPI. In particular, I highlight three lessons from the formula. The first two lessons come from each of the two components of the formula. The formula is a weighted sum of different log prices in different sectors, where the weight represents the welfare cost of inflation in each sector. I show that the weight consists of two parts, one representing the size of the sector and the other representing the sensitivity of the production wedge to inflation in the sector. I also show that the RPI formula does not directly depend on international prices. The third lesson comes from what is not in the formula.

The first lesson from the first component of the RPI is that the size of the sector in the RPI weight needs to be measured in terms of the production size rather than the consumption size. This is because the cost of inflation in my model is the efficiency loss in production. If there is inefficiency in production, it is welfare damaging either through reduced consumption, more work or a negative effect on the trade balance, which affects the economy through a tighter budget constraint. Therefore, regardless of whether its output is consumed or exported, inefficiency in production is costly in a sector that is large in terms of production. An implication of this is that the central banks should stabilize PPI rather than CPI if everything else is constant. However, there is a caveat in this simple takeaway, as my quantitative analysis shows that the stabilization of PPI does not necessarily perform better than CPI stabilization due to the second component of the RPI weight.

The second component of the RPI weight is a combination of a well-known stickiness parameter and less frequently highlighted but equally important parameter, representing the elasticity of substitution between differentiated goods within a sector. These two parameters govern the sensitivity of inefficiency to inflation in the sector in question. The mechanism comprises two steps. First, volatile inflation causes mispricing by the firms in a sector. This step depends on the degree of price stickiness. Second, mispricing leads to deviations of demand and production from the efficient level. This step depends on the elasticity of substitution.

The addition of sectoral heterogeneity in the elasticity of substitution provides the second lesson that is important when we discuss core inflation targeting versus headline inflation targeting. Recall that the difference between the two measures is whether they include commodity prices such as food and energy[2]. While the literature to date has focused on one characteristic of commodities, namely price flexibility, the high elasticity of substitution is also an important characteristic[3]. As is standard in the conventional argument, if we base our decision only on the price flexibility of different sectors, we should assign a lower weight to commodity sectors and thus favor the use of core inflation targeting. However, if we focus on the latter characteristic, we should place greater weight on commodity sectors. Given my analytical formula, whether we should place less weight on prices in commodity sectors or not depends on the relative size of price flexibility and elasticity.

The third lesson from the analytical formula is that exogenous international prices do not appear in it. This is despite the fact that I naturally model the effect of exogenous international prices. In my model, the firms respond to the change in the cost of imported material caused by the change in the international price of inputs. The firms also know that a deviation of their export price from those of their international competitors results in a change in export demand. I show that these international prices affect the optimal price index if and only if they affect the output prices of domestic sectors. This is because volatile inflation causes efficiency loss in production regardless of the cause of the volatility, and thus, we do not need to adjust the formula for the price index depending on whether such volatility comes from international prices.

As an implication, although we may tend to think that central banks are not responsible for inflation volatility caused by international price movements, a central bank should be concerned about volatility as long as it affects the RPI. To understand this point, note that although international prices are exogenous, domestic prices can be controlled via changes in the exchange rate. Imagine an economy where all the domestic prices of different sectors are proportional to the international prices in those sectors. The ratio between the vector of international prices and the vector of domestic prices is the exchange rate. If the central bank selects one domestic sector, it is possible to stabilize the domestic price of that sector by adjusting the exchange rate to offset international price movements. Of course, this operation affects all other sectors, so the central bank faces a trade-off between stabilizing one sector and stabilizing another. The RPI indicates how to balance this trade-off.

My second main result is obtained from quantitative analysis, where I compare the welfare under simple stabilization policies for the RPI and three conventional price indices. Here, a

[2]Although the original definition of the core inflation rate involves econometric models that attempt to identify the persistent component of the inflation rate (see, for example, Wynne 2008), the optimal monetary policy literature has practically interpreted the core index as an index excluding food and energy.

[3]See Nakamura and Steinsson [2008] on price flexibility and Broda and Weinstein [2006]on the elasticity of substitution.

simple stabilization policy means a policy in which the inflation rate in terms of the price index in question is zero in both the short and long run. In reality, implementing these policies via either Taylor rules or exchange rate interventions is simpler than implementing the Ramsey solution itself. However, it is not obvious that the simple stabilization of the RPI yields higher welfare than the stabilization of other price indices since the analytical result only states the optimality of long-run stabilization of the RPI, and the Ramsey solution itself, in general, involves short-run deviations from complete stabilization.

Calibrating to 40 countries with 35 sectors, I show that, for all countries in my sample, RPI stabilization performs the best among the stabilization schemes for the four indices considered. The loss from a simple stabilization of the RPI compared with the Ramsey solution turns out to be negligible and less than one-hundredth, on average, of the loss from simple stabilization of the other indices in terms of steady-state consumption. This means that the RPI is suitable not only for long-run stabilization targets but also for short-run targets.

Another important point from the welfare calibration is that there is no simple takeaway other than the RPI. This is because the ranking of other stabilization policies varies across countries depending on the combination of trade patterns and price stickiness. That is, CPI targeting performs better than PPI targeting for some countries while headline CPI performs better than core CPI targeting for other countries, depending on the combination of price stickiness, the elasticity of substitution, and trade patterns. The only result common to all countries in my sample is that RPI stabilization performs better than the stabilization of the other indices.

## 1.1 Related literature

This paper is an open economy extension of the method to derive the optimal price index from the Ramsey problem developed in Woodford [2010]. The price index in Woodford [2010] can be obtained as a special case of the RPI proposed in this paper by letting the exports in each sector be zero and requiring the elasticity of substitution to be homogeneous across sectors. However, the other direction, i.e., deriving the RPI from Woodford's index, is not straightforward. This is because the size of each sector in Woodford [2010] can be interpreted either as the size of consumption or the size of production, and one might suggest different open economy extensions of the index depending on the interpretation. My analysis and the resulting formula for the RPI show that the correct interpretation is the size of production.

This paper is the first to theoretically show that the size of sectors in the stabilization objective should be measured by production size rather than consumption size in a multi-sector SOE environment. A similar feature can be seen in the result of Gali and Monacelli [2005], who demonstrate the optimality of output price stabilization in a model with only one production sector. However, having multiple sectors is key to answering the question of which price index to target since this creates the crucial trade-off between stabilizing one sector versus another when the first-best allocation cannot be achieved. In particular, their analysis cannot tell whether the result is coming from the assumption that there is only one sector with sticky prices or the assumption that the economy produces in only one sector. This makes it difficult to generalize their model to various trade patterns commonly observed in the real world such as the commodity importing case. My general formula enables me to separately discuss the effect of production and stickiness and can be applied not only to the special case of Gali and Monacelli [2005] but also to the opposite polar case (commodity exporter) and the intermediate cases.

There is a literature that analyzes the optimal monetary policy in two-country models (see Corsetti et al. 2010 and Engel 2011, for examples) and the models of a monetary union (see Gali and Monacelli [2008] and Kekre [2018], for examples). This paper differs from this literature in two senses. First, although, similarly to Woodford [2010] and this paper, these papers often identify the central bank's trade-off depending on price stickiness, they do not derive the price index that balances the trade-off except for special cases that achieve the first-best allocation. Second, the two-country setups of these papers are essentially closed since the two countries (or the countries in the union) do not trade with the rest of the world. Therefore, their framework cannot answer the question of how to deal with international price movements.

In this paper, I use the term "optimal price index", but the derived price index does not necessarily coincide with the optimal indices in the literature on index theory: see Diewert et al. [2009], for example. This is because the purposes of the index are different. In index theory, Diewert et al. [2009] among others attempt to obtain an accurate measure of the cost of living while my aim is to obtain the index for the central bank's stabilization target. By solving the household's optimization condition in the partial equilibrium sense, we can see that the CPI is the optimal price index in the sense of the cost of living in my model. However, my analysis shows that the optimal price index for the central bank's stabilization target is different from the CPI. It is natural to obtain different optimal price indices for different purposes.

From a technical point of view, the open economy extension in this paper involves two innovations that are also applicable to other SOE problems. The first is the definition of the Ramsey problem, which is consistent with the assumption of the timing of asset markets. Specifically, the Ramsey planner needs to recognize that some of the effects of its policy will be offset by the insurance effect of the asset market. In this way, I can compare the central bank's second-best problem with the planner's first-best problem and offer intuitive discussions comparing the two. The definition of the Ramsey problem is in line with the Ramsey taxation literature, but the previous NK SOE literature has defined the Ramsey problem in a different way, and hence, the first-best allocation cannot serve as a benchmark for the analysis. The definition of the Ramsey problem in this paper can simplify and clarify the analysis by De Paoli [2009], for example, of the case of the inefficient steady state.

The second innovation of this paper is differential tax rates that depend on the place of consumption, which allows me to simplify the analysis under terms of trade externalities without relying on extreme assumptions on parameter values. This is another feature that distinguishes my paper from Gali and Monacelli [2005], who impose a subsidy that partially offsets steady-state inefficiency and eliminate the rest of inefficiency by setting a parameter value such that the value of exports does not respond following any shock. I believe my novel simplification is useful for monetary policy discussions under terms of trade externalities.

The remainder of the paper proceeds as follows. In Section 2, I first explain the SOE NK DSGE model with which I define the Ramsey problem. In Section 3, I explain my analytical results. I first state the key assumptions on tax rates that make the analysis simple before approximating the Ramsey problem. The main theorem states that the RPI is stabilized in the long run, which is the justification for my proposal of RPI stabilization. Section 4 discusses the quantitative welfare comparison. Section 5 concludes the paper.

# 2 Method

I derive the RPI by solving the Ramsey problem of a central bank attempting to maximize the welfare of a representative household given market constraints in an SOE NK DSGE model. This section describes these market constraints and defines the Ramsey problem.

The economy features an arbitrary number of sectors with heterogeneous output price stickiness a la Calvo [1983]. There is no domestic input-output structure, but the production requires labor and imported intermediate goods. The output can either be exported or domestically consumed. When exported, the price is sticky in the producer currency. Specifically, I denote the number of sectors by $S \in \mathbb{N}$, within each of which, a continuum of firms produce differentiated goods. The differentiated goods are aggregated within each sector.

The economy is small and open in the sense that international conditions are exogenous. The costs of imported materials are given by the exogenous international price times the endogenous exchange rate. The price of exports is compared with the exogenous prevailing price in the international market, to which the foreign demand for the country's export responds. The economy also takes the asset prices in complete international asset markets as given.

The monetary authority attempts to maximize the welfare of the representative domestic household, which consumes goods from all the sectors and provides labor. The monetary authority takes the optimization behavior of the household and firms under staggered price setting as given. It also takes exogenous international market conditions as given. I assume the timeless perspective following Woodford [2003].

## 2.1 Market conditions

Sectors are heterogeneous in price stickiness and the elasticity of substitution across differentiated goods within a sector. The former is already identified as key to obtaining the optimal price index in the closed economy literature. Although heterogeneity in the elasticity has not been highlighted in the literature, it is quantitatively important and intuitive. That is, a high elasticity of substitution implies that a small mispricing leads to a tremendous swing in demand and is thus costly to welfare.

For the model to be applicable to different countries with different trade patterns, I use a general production technology and a general trade pattern. By adjusting the parameter of the production technology of my model, one can consider a country such as Japan importing commodities, i.e., goods with flexible prices and high elasticities of substitution, and exporting differentiated goods or a country such as Russia doing the opposite.

Compared to the common SOE framework featuring tradable goods and non-tradable goods or that with home goods and foreign goods, the description of the production sector is enriched such that any imported good goes through the domestic sector before being consumed by the household. This allows me to treat different sectors uniformly despite the generality. My model encompasses the common frameworks in the literature as special cases.

### 2.1.1 The representative household

In any period of time $t \in [0, \infty]$, the representative household consumes goods from each of the $S$ sectors denoted by $C_{st}$ for $s \in S$ and supplies labor, denoted by $L_{st}$, to each of the $S$ sectors. I assume that the amounts of consumption from different sectors are aggregated in a Cobb-Douglas function with the exponential factor $\psi_s$ for sector $s \in S$ summing up to one $\sum_{s \in S} \psi_s = 1$.

$$C_t = \prod_{s \in S} C_{st}^{\psi_s} \tag{1}$$

This implies that elasticity of substitution across sectors is one. This is the standard assumption used in multi-sector NK models; see, for example, Aoki [2001] and Eusepi et al. [2011].[4] For the labor supply, I simply assume homogeneous labor that can be summed. This means that the disutility from labor depends only on the aggregate amount of work, not in the distribution of where the household works.

$$L_t = \sum_{s \in S} L_{st} \tag{2}$$

An alternative would be to assume increasing disutility from labor supplied to each firm in each sector. This would increase the efficiency cost of price dispersion relative to my case.

Given prices $\{P_{st}\}_{s \in S}, W_t$, profits $\{E_{st}\}_{s \in S}$, a lump sum transfer $T_t$, all denominated in the local currency, the pricing kernel in the international asset market $\mathcal{M}_t^*$ , the exchange rate $\mathcal{E}_t$, and the price $\Lambda$ of initial debt $D_0$, where the unit is in the utility in the pre-specified insurance contract over different policies, the household maximizes

$$\max_{D_0, \{C_{st}, L_{st}\}_{s \in S, t \in [0,\infty]}} E_0 \sum_{t=0}^{\infty} \beta^t \left[ \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\phi}}{1+\phi} \right] + \Lambda D_0,$$

subject to

$$E_0 \sum_{t=0}^{\infty} \frac{\mathcal{M}_{0t}^*}{\mathcal{E}_t} \left( \sum_{s \in S} W_t L_{st} + \sum_{s \in S} E_{st} + T_t - \sum_{s \in S} P_{st} C_{st} \right) \geq D_0. \tag{3}$$

The first-order conditions are as follows:

$$\beta^t \psi_s \frac{C_t^{1-\sigma}}{C_{st}} = \frac{\mathcal{M}_{0t}^*}{\mathcal{E}_t} \lambda P_{st}$$
$$\beta^t L_t^{\phi} = \frac{\mathcal{M}_{0t}^*}{\mathcal{E}_t} \lambda W_t$$
$$\Lambda = \lambda.$$

The first term $\left(\prod_{s \in S} C_{st}^{\psi_s}\right)^{1-\sigma} / (1-\sigma)$ in the objective function represents the instantaneous utility from consumption from each sector $\{C_{st}\}_{s \in S}$ aggregated according to $C_t =$

[4] This does not mean that the assumption is without loss of generality. Benigno and Benigno [2003], for example, demonstrate that relaxing the assumption of a unitary elasticity between a home good and a foreign good may change the desirability of the flexible price allocation.

$\prod_{s \in S} C_{st}^{\psi_s}$. The second term in the objective function represents the disutility from labor supply to each sector $\{L_{st}\}_{s \in S}$. From the expenditure minimization problem, the CPI consistent with this consumption aggregator is

$$P_t = \prod_{s \in S} \left( \frac{P_{st}}{\psi_s} \right)^{\psi_s}. \tag{4}$$

Using this, intra-temporal conditions for the household's optimization are expressed as follows:

$$\psi_s C_t = \frac{P_{st}}{P_t} C_{st}, \ \forall s \in S \tag{5}$$

$$\frac{L_t^{\phi}}{C_t^{-\sigma}} = \frac{W_t}{P_t}. \tag{6}$$

I assume that the household trades in the international asset market before the monetary authority chooses its policy. With this timing convention, the marginal utility for the household of having less debt $D_0$ is fixed at the exogenous level $\Lambda$ across different possible monetary policies. The constant $\Lambda$ represents the shadow price of the initial debt in the asset markets. This allows me to subsequently derive an international risk sharing condition that is invariant across policies. The policy-invariant risk sharing condition is standard in the literature, but how to consistently derive the condition in a DSGE setup has not been fully explored. For further discussion, see Senay and Sutherland [2007].

The level of consumption is determined by the tightness of the lifetime budget constraint. Denoting the aggregate consumption of a foreign country and its price by $C_t^*$and $P_t^*$, we can consider the stochastic discount factor to be equated to the ratio of marginal utilities of the consumer in that foreign country between any two states of the world. In particular, if we let $\mathcal{M}_{0,t}^* = \prod_{\tau=1}^{t} \mathcal{M}_{\tau}^*$ be the discount factor from period 0, or the planning period, to period $t$ in the future, then, assuming the same utility function for the foreign consumer consuming $C_t^*$ at price $P_t^*$, we can interpret the stochastic discount factor as

$$\mathcal{M}_{0,t}^* = \beta^t \frac{(C_t^*)^{-\sigma} / P_t^*}{(C_0^*)^{-\sigma} / P_0^*} \tag{7}$$

under the assumption that the foreign consumer also has access to the same complete asset markets. Gali and Monacelli [2005] also interpret the stochastic discount factor in this way. Combining this with the inter-temporal condition of the household, we have

$$\beta^t \frac{(C_t^*)^{-\sigma} / P_t^*}{(C_0^*)^{-\sigma} / P_0^*} \Lambda = \beta^t C_t^{-\sigma} \mathcal{E}_t P_t^{-1}.$$

Thus, we can obtain the international risk sharing condition

$$C_t = \xi C_t^* Q_t^{\frac{1}{\sigma}}, \tag{8}$$

where $Q_t = \mathcal{E}_t P_t^* / P_t$ is the real exchange rate and $\xi = (\Lambda P_0^*)^{-\frac{1}{\sigma}} / C_0^*$ is a constant. For this SOE, foreign consumption $C_t^*$ and the foreign consumption price level $P_t^*$ are exogenous, so is the stochastic discount factor $\mathcal{M}_t^*$. Note that if we do not assume the asset markets that insure across different policies, we need to allow $\Lambda$ to vary across policies and hence the coefficient of the risk sharing condition also varies across policies.

### 2.1.2 The individual firm's technology and aggregation

The production technology for firm $i$ in sector $s \in S$ is given by

$$Y_{sit} + Y_{sit}^X = Z_{s,t} M_{sit}^{\alpha_{sm}} L_{sit}^{\alpha_{sl}}.$$

$Y_{sit}$ and $Y_{sit}^X$ are the output of firm $i$ in sector $s$ at time $t$ shipped for domestic use and exported to foreign, respectively, $Z_{s,t}$ is the stochastic sector-specific productivity, $M_{sit}$ is the imported good, and $L_{sit}$ is labor. Note that the Cobb-Douglas parameters $\alpha_{sm}$ and $\alpha_{sl}$ are allowed to vary across sectors.

I assume that the technology is linear, that is, $\alpha_{sm} + \alpha_{sl} = 1$ for all $s \in S$. When $\alpha_m = 0$, this reduces to the production technology assumed in Gali and Monacelli [2005]. The linear technology assumption makes the following calculation simpler by making the marginal cost independent of the amount produced. If one instead assumes decreasing returns to scale, the efficiency cost of price dispersion will be larger. For simplicity, I also assume $\alpha_{sl} > 0$ for all $s \in S$. This means that all sectors use at least some amount of labor. This is empirically true. Some countries, on the other hand, may import nothing in some sectors. Therefore, I do not impose $\alpha_{sm} > 0$.

By setting $\alpha_{sm} \approx 1$ and $\alpha_{sl} \approx 0$, I can consider a country importing in sector $s$. Alternatively, by setting $\alpha_{sm} \approx 0$ and $\alpha_{sl} \approx 1$, I can consider a country being skilled at producing goods in sector $s$, and depending on the demand from foreign, it is likely that the country exports in sector $s$ in equilibrium.

There is an aggregation firm in each sector with aggregation technologies

$$Y_{st} = \left( \int Y_{sit}^{\frac{\theta_s - 1}{\theta_s}} di \right)^{\frac{\theta_s}{\theta_s - 1}} \quad \text{and} \quad Y_{st}^X = \left( \int \left( Y_{sit}^X \right)^{\frac{\theta_s - 1}{\theta_s}} di \right)^{\frac{\theta_s}{\theta_s - 1}}, \tag{9}$$

that operates competitively. The elasticity of substitution parameter $\theta_s$ can be heterogeneous across sectors. The cost minimization problem of the aggregator gives the demand schedule

$$Y_{sit} = \left( \frac{P_{sit}}{P_{st}} \right)^{-\theta_s} Y_{st} \quad \text{and} \quad Y_{sit}^X = \left( \frac{P_{sit}^X}{P_{st}^X} \right)^{-\theta_s} Y_{st}^X, \tag{10}$$

and the price index consistent with the aggregation

$$P_{st} = \left( \int P_{sit}^{1-\theta_s} di \right)^{\frac{1}{1-\theta_s}} \quad \text{and} \quad P_{st}^X = \left( \int \left( P_{sit}^X \right)^{1-\theta_s} di \right)^{\frac{1}{1-\theta_s}}. \tag{11}$$

Note that the output for domestic use and foreign export are the same goods but labeled and priced differently.

### 2.1.3 The individual firm's pricing decision

Assume that in each sector $s \in S$, a randomly selected fraction $1 - \lambda_s$ of the firms can reset the price. The price stickiness parameter $\lambda_s$ can also vary across sectors. An individual firm in sector $s$ takes wage $W_t$, import price $\mathcal{E}_t Q_{st}^*$, the demand function in equations (10), production function and tax $\tau_s$ as given. The unit cost of imported good $\mathcal{E}_t Q_{st}^*$ is given by the product of the endogenous exchange rate $\mathcal{E}_t$ and exogenous and stochastic international price $Q^*_{st}$. The prices of its output are set by the individual firm to maximize its expected profit.

$$\begin{aligned}\left(P_{sit}\left(0\right),P^X_{sit}\left(0\right)\right) = \arg\max_{\left(P,P^X\right)} \sum_{\tau=0}^{\infty} \lambda_s^{\tau} E_t \Bigg[ \frac{\mathcal{E}_t}{\mathcal{E}_{t+\tau}} \mathcal{M}^*_{t,t+\tau} \\ \times \Bigg\{ \left( \left(1-\tau_s\right) P - \left( \frac{\mathcal{E}_{t+\tau} Q^*_{s,t+\tau}}{\alpha_{sm}} \right)^{\alpha_{sm}} \left( \frac{W_{t+\tau}}{\alpha_{sl}} \right)^{\alpha_{sl}} Z^{-1}_{s,t+\tau} \right) \left( \frac{P}{P_{s,t+\tau}} \right)^{-\theta_s} Y_{s,t+\tau} \\ + \left( \left(1-\tau^X_s\right) P^X - \left( \frac{\mathcal{E}_{t+\tau} Q^*_{s,t+\tau}}{\alpha_{sm}} \right)^{\alpha_{sm}} \left( \frac{W_{t+\tau}}{\alpha_{sl}} \right)^{\alpha_{sl}} Z^{-1}_{s,t+\tau} \right) \left( \frac{P^X}{P^X_{s,t+\tau}} \right)^{-\theta_X} Y^X_{s,t+\tau} \Bigg\} \Bigg] \end{aligned} \tag{12}$$

The realized profit $E_{sit}$ is aggregated within and across sectors $E_{st} = \int E_{sit} di$ and immediately paid out to the household. Note that the firms are taxed differently across sectors and between destinations. The rate for profits earned domestically is $\tau_s$ and the rate for profits from foreign is $\tau^X_s$.

Following the usual procedure, the optimal pricing condition can be aggregated to

$$\frac{P_{s,t}}{P_t} = \frac{P_{s,t-1}}{P_{t-1}} \frac{1}{\Pi_t} \left( \frac{1}{\lambda_s} + \left(1 - \frac{1}{\lambda_s}\right) \left( \frac{\tilde{F}_{s,t}}{\tilde{K}_{s,t}} \right)^{\theta_s - 1} \right)^{\frac{1}{\theta_s - 1}} \tag{13}$$

$$\frac{P^X_{s,t}}{P_t} = \frac{P^X_{s,t-1}}{P_{t-1}} \frac{1}{\Pi_t} \left( \frac{1}{\lambda_s} + \left(1 - \frac{1}{\lambda_s}\right) \left( \frac{\tilde{F}^X_{s,t}}{\tilde{K}^X_{s,t}} \right)^{\theta_s - 1} \right)^{\frac{1}{\theta_s - 1}} \tag{14}$$

where $\tilde{F}_{s,t}, \tilde{K}_{s,t}, \tilde{F}^X_{s,t}, \tilde{K}^X_{s,t}$ are defined as follows:

$$\tilde{F}_{s,t} = C_t^{-\sigma} \frac{P_{s,t}}{P_t} Y_{s,t} + \lambda_s \beta E_t \left(\Pi_{s,t+1}\right)^{\theta_s - 1} \tilde{F}_{s,t+1} \tag{15}$$

$$\tilde{K}_{s,t} = \left(1-\tau_s\right)^{-1} \frac{\theta_s}{\theta_s - 1} C_t^{-\sigma} \left( \frac{Q_t Q^*_{s,t}}{\alpha_{sm} P^*_t} \right)^{\alpha_{sm}} \left( \frac{W_t}{\alpha_{sl} P_t} \right)^{\alpha_{sl}} Z^{-1}_{s,t} Y_{s,t} + \lambda_s \beta E_t \left(\Pi_{s,t+1}\right)^{\theta_s} \tilde{K}_{s,t+1} \tag{16}$$

$$\tilde{F}^X_{s,t} = C_t^{-\sigma} \frac{P^X_{s,t}}{P_t} Y^X_{s,t} + \lambda_s \beta E_t \left(\Pi^X_{s,t+1}\right)^{\theta_s - 1} \tilde{F}^X_{s,t+1} \tag{17}$$

$$\tilde{K}^X_{s,t} = \left(1-\tau^X_s\right)^{-1} \frac{\theta_s}{\theta_s - 1} C_t^{-\sigma} \left( \frac{Q_t Q^*_{s,t}}{\alpha_{sm} P^*_t} \right)^{\alpha_{sm}} \left( \frac{W_t}{\alpha_{sl} P_t} \right)^{\alpha_{sl}} Z^{-1}_{s,t} Y^X_{s,t} + \lambda_s \beta E_t \left(\Pi^X_{s,t+1}\right)^{\theta_s} \tilde{K}^X_{s,t+1} \tag{18}$$

Note that the nominal exchange rate is substituted out using the definition of the real exchange rate $Q_t = \mathcal{E}_t P^*_t / P_t \Leftrightarrow \mathcal{E}_t = Q_t P_t / P^*_t$, and I defined CPI inflation rate as $\Pi_t = P_t / P_{t-1}$ and sectoral inflation rates as $\Pi_{s,t} = P_{st}/P_{st-1}, \Pi^X_{s,t} = P^X_{st}/P^X_{st-1}$. For the derivation, see Appendix A.1.

Equations (13) and (14) govern the dynamics of sectoral inflation. Note that the sectoral inflation rate $\Pi_{s,t}$ and the inflation in terms of the CPI $\Pi_t$ are related through the change in the relative price $P_{st}/P_t$. Thus, the equations state that sectoral inflation is a function of expected future sectoral inflation $\tilde{F}_{s,t}$ and the expected future marginal cost $\tilde{K}_{s,t}$. The sectoral inflation rate is the weighted sum of one and the ratio $\tilde{F}_{s,t}/\tilde{K}_{s,t}$, where the weight on one becomes larger as the price becomes stickier $\lambda_s \to 1$. When the price is completely sticky $\lambda_s = 1$, then sectoral inflation becomes one, meaning that the nominal sectoral price is fixed at the previous level, and only the relative price may move if the CPI $P_t$ moves. At the other extreme, when the price is fully flexible $\lambda_s \to 0$, these equations hold by having $\tilde{F}_{st} = \tilde{K}_{st}$. In this case, the expectation terms in $\tilde{F}_{st}$ and $\tilde{K}_{st}$ also disappear, restoring the flexible price equilibrium pricing rule

$$\frac{P_{s,t}}{P_t} = (1-\tau_s)^{-1} \frac{\theta_s}{\theta_s - 1} \left(\frac{Q_t Q_{s,t}^*}{\alpha_{sm} P_t^*}\right)^{\alpha_{sm}} \left(\frac{W_t}{\alpha_{sl} P_t}\right)^{\alpha_{sl}} Z_{s,t}^{-1}.$$

### 2.1.4 Resource constraints

The market clearing conditions are

$$\sum_{s \in S} \int L_{sit} di = L_t, \int M_{sit} di = M_{st}, C_{st} = Y_{st}, X_{st} = Y_{st}^X.$$

Using the factor demand from individual firms, these reduce to market clearing conditions in aggregate variables

$$C_{st} = Y_{st} \qquad \text{and} \qquad X_{st} = Y_{st}^X \tag{19}$$

and the resource constraints in aggregate variables

$$Z_{st} L_{st} = \left(\frac{\alpha_{sl}}{\alpha_{sm}} \frac{Q_t Q_{st}^* / P_t^*}{W_t / P_t}\right)^{\alpha_{sm}} \left(\Delta_{st} C_{st} + \Delta_{st}^X X_{st}\right) \quad \text{and} \quad M_{st} = \frac{\alpha_{sm}}{\alpha_{sl}} \frac{W_t / P_t}{Q_t Q_{st}^* / P_t^*} L_{st}, \tag{20}$$

where $\Delta_{st} = \int \left(\frac{P_{sit}}{P_{st}}\right)^{-\theta_s} di \geq 1$ and $\Delta_{st}^X = \int \left(\frac{P_{sit}^X}{P_{st}^X}\right)^{-\theta_s} di \geq 1$ are the production wedges that evolve according to

$$\Delta_{st} = \lambda_s \left(\frac{P_{st}}{P_{st-1}}\right)^{\theta_s} \Delta_{s,t-1} + (1-\lambda_s) \left(f_s\left(\frac{P_{s,t}}{P_t}, \Pi_t; \frac{P_{s,t-1}}{P_{t-1}}\right)\right)^{\theta_s} \tag{21}$$

$$\Delta_{st}^X = \lambda_s \left(\frac{P_{st}^X}{P_{st-1}^X}\right)^{\theta_s} \Delta_{s,t-1}^X + (1-\lambda_s) \left(f_s\left(\frac{P_{s,t}^X}{P_t}, \Pi_t; \frac{P_{s,t-1}^X}{P_{t-1}}\right)\right)^{\theta_s}, \tag{22}$$

where the function $f_s$ is defined as

$$f_s(x, y; z) = \left(\frac{1}{1-\lambda_s}\left(1 - \lambda_s \left(\frac{xy}{z}\right)^{\theta_s - 1}\right)\right)^{\frac{1}{\theta_s - 1}}.$$

For the derivation, see Appendix A.2.

Equation (20) combined with the dynamics (21) and (22) are the key equations capturing the cost of inflation in sector $s$.

First, as we can see from the dynamics, sectoral inflation or deflation $\Pi_{st} = P_{st}/P_{st-1}$ causes larger wedges $\Delta_{st}, \Delta_{st}^X$. When sectoral inflation is zero, i.e., $\Pi_{st} = 1$, the wedge decays at the rate $\lambda_s$ to the steady state of $\Delta_{st} = 1$. When the inflation rate deviates from one, it enlarges the deviation of the wedge from one.[5] The effect of inflation on the wedge is larger

[5] This happens regardless of inflation or deflation. The first term is increasing in $\Pi_{st} = P_{st}/P_{st-1}$, but the second term is decreasing in $\Pi_{st} = (P_{s,t}/P_t)\,\Pi_t / (P_{s,t-1}/P_{t-1})$. The overall term behaves like the first term when $\Pi_{st} \gg 1$ and like the second term when $\Pi_{st} \ll 1$.

when the price is sticky, represented by a larger $\lambda_s$, and when the differentiated goods are more substitutable, represented by a larger $\theta_s$. Price stickiness limits the ability of firms to set a uniform price across differentiated goods. A higher elasticity induces a larger response of demand and thus production to the price differential among similar goods within the sector.

Second, the aggregate resource constraint (20) states that the wedges $\Delta_{st}, \Delta^X_{st}$ create a gap between the input $L_{st}$ and the outputs $C_{st}, X_{st}$ in effective units, which is the ultimate source of welfare loss in my model. Even if the production function in each firm is not affected by the inflation rate, the distribution of production within the sector is affected by inflation, as explained in the previous paragraph. Since uneven outputs are translated into a lower effective output under the love of variety assumption represented by the CES aggregator (9), sectoral inflation causes the production wedges.

### 2.1.5 Small open economy assumptions

Finally, I assume that foreign demand is price elastic.

$$X_{st} = \left( \frac{P^X_{st}}{\mathcal{E}_t P^*_{st}} \right)^{-\theta^*_s} X^*_{st}, \tag{23}$$

where $X^*_{st}$ is the exogenous total international demand for sector $s$ and $P^*_{st}$ is its aggregate price index that is also exogenously given. This assumption can be derived from the cost minimization condition of a foreign buyer who aggregates the composite goods of sector $s$ from different countries with a constant elasticity of substitution $\theta^*_s$ aggregator.

## 2.2 The Ramsey problem

The monetary authority's problem is defined as follows.

**Definition 1.** The optimal monetary policy is the solution to the following problem. Given random shocks $\left( (Q^*_{st}/P^*_t, P^*_{st}/P^*_t, Z_{st}, X^*_{st})_{s\in S}, C^*_t \right)_{t=0}^{\infty}$, tax $\left(\tau_s, \tau^X_s\right)_{s\in S}$, and initial state variables $P_{-1}$, $\mathcal{E}_{-1}$, $\left(\Delta_{s,-1}, \Delta^X_{s,-1}\right)_{s\in S}$ the central bank chooses a contingent plan of all the endogenous variables $C_t$, $L_t$, $\left(C_{st}, L_{st}, P_{st}/P_t, P^X_{st}/P_t, Y_{st}, Y^X_{st}, X_{st}, M_{st}\right)_{s\in S}$, $\frac{W_t}{P_t}$, $Q_t$, $\Pi_t$, $\left(\Delta_{st}, \Delta^X_{st}\right)_{s\in S}$, $\left(\tilde{K}_{s,t}, \tilde{F}_{s,t}, \tilde{K}^X_{s,t}, \tilde{F}^X_{s,t}\right)_{s\in S}$, $D_0$ to solve

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \left[ \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\phi}}{1+\phi} \right] + \Lambda D_0$$

subject to equations (1), (2), (4)-(6), (8), (13)-(23) and

$$E_0 \sum_{t=0}^{\infty} \left[ \mathcal{M}^*_{0,t} P^*_t \sum_{s\in S} \left( X_{st} \frac{P^X_{st}}{Q_t P_t} - M_{st} \frac{Q^*_{st}}{P^*_t} \right) \right] = D_0.$$

The last condition is equivalent to the household's lifetime budget constraint (3) under the assumption that all the profit goes to the household as $E_t$ and the balanced government budget. This condition is important for binding the planner with the same trade-off between consumption and labor as that faced by the decentralized economy.

Although the initial level of debt $D_0$ is mathematically expressed as a choice variable, this does not mean that the central bank can freely choose it. Recall that I assumed in the previous sub-section that the asset markets operate before the monetary authority chooses its policy. Thus, the monetary authority takes into account the change or lack thereof in the initial level of debt $D_0$ when it chooses its policy. In this sense, the monetary authority indirectly chooses the initial level of debt.

# 3 Analytical Results

In this section, I derive the formula for the RPI and discuss the intuition behind the index. The justification of the index is given in a theorem that states that RPI needs to remain constant in long-run expectation for the economy to achieve the Ramsey optimal allocation. I start by showing two lemmas that help us understand the trade-off faced by the central bank.

The first lemma concerns the steady-state property that makes the analysis tractable. The second lemma shows how the Ramsey problem can be approximated around the steady state. As studied in Benigno and Woodford [2012], the solution to the approximated problem approximates the solution to the original Ramsey problem under regularity conditions.

Then, I state the theorem on the optimality of stabilizing the RPI. The formula for RPI can be interpreted as a weighted sum of prices in different sectors, where the weight depends on output share of the sector, price stickiness and the elasticity of substitution within the sector. I discuss two points on the formula. First, compared with the CPI, the RPI is closer to PPI since PPI includes prices of exports. However, the PPI is not always better than CPI due to the other two factors: price stickiness and the elasticity of substitution. Second, international prices do not directly appear in the formula. This means that the central bank should be concerned about international prices if and only if they affect output prices that appear in the RPI formula.

## 3.1 Terms of trade externality and the efficiency of the steady state

To focus on the monetary friction in the analysis, it is convenient to assume that the tax rates are set to offset any real distortions that arise under the flexible price equilibrium. There are two types of real distortions in this economy: monopolistic distortions and terms of trade externality. It is widely known what tax rate offsets the former since it also arises in the closed economy setup. Regarding the latter, however, no paper has explicitly defined the distortion and offset it using a tax.

In this subsection, I show that these distortions can be offset by taxes if we assume different tax rates between domestic consumption and exports, as I do in my model. The distortions are defined as wedges between the social planner's allocation and the flexible price equilibrium. The planner's problem is defined as the maximization of the household's welfare subject only to the resource and technology constraint and the conditions in international markets. The flexible price equilibrium is defined as usual. Monopolistic competition leads to monopolistic markups in the price that appear as distortions in the allocation. The terms of trade externality, on the other hand, comes from the inability of the individual firms to exploit monopolistic competition in the international market.

I define the first-best planner's problem as follows.

**Definition 2.** Given $\left(\left(\frac{Q^*_{st}}{P^*_t}, \frac{P^*_{st}}{P^*_t}\right)_{s\in S}, \mathcal{M}^*_{0,t}\right)_{t=0}^{\infty}, \Lambda$, the planner solves

$$\max_{D_0,\left((C_{st},M_{st},X_{st},L_{st})_{s\in S}\right)_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \left[ \frac{\left(\prod_{s\in S} C_{st}^{\psi_s}\right)^{1-\sigma}}{1-\sigma} - \frac{\left(\sum_{s\in S} L_{st}\right)^{1+\phi}}{1+\phi} \right] + \Lambda D_0,$$

subject to the technology constraint

$$Z_{s,t} M_{st}^{\alpha_{sm}} L_{st}^{\alpha_{sl}} = C_{st} + X_{st} \;\forall s \in S$$

and the inter-temporal trade balance condition

$$E_0 \sum_{t=0}^{\infty} \left[ \mathcal{M}^*_{0,t} P^*_t \sum_{s\in S} \left( X_{st}^{\frac{\theta^*_s - 1}{\theta^*_s}} X_{st}^{*\frac{1}{\theta^*_s}} \frac{P^*_{st}}{P^*_t} - \frac{Q^*_{st}}{P^*_t} M_{st} \right) \right] = D_0.$$

In defining the planner's problem, I use

$$X_{st} = \left( \frac{P^X_{st}}{\mathcal{E}_t P^*_{st}} \right)^{-\theta^*_s} X^*_{st} \Leftrightarrow \frac{P^X_{st}}{\mathcal{E}_t P^*_{st}} = \left( \frac{X_{st}}{X^*_{st}} \right)^{-\frac{1}{\theta^*_s}}$$

to eliminate prices.

The objective function is the same as the welfare of the household in the Ramsey problem in Definition 1. The first-best planner is constrained only by the aggregate production technology in each sector and the inter-temporal trade balance condition. In building the aggregate production function, I already imposed uniform production within a sector $Y_{sit} = Y_{st}$ and so forth, as the optimality condition. The inter-temporal trade balance condition does not necessarily require balanced trade in each period, but any trade deficit is financed in the international asset market, and any trade surplus is invested in the international asset market such that the discounted sum of the trade surplus equals the initial level of the external debt $D_0$.

Appendix B.1 shows that the planner's solution is characterized by the following:

$$C_t \frac{\psi_s}{C_{st}} \alpha_{sl} Z_{s,t} \left( \frac{\alpha_{sm}}{\alpha_{sl}} \frac{L_t^{\phi}}{C_t^{-\sigma}} \frac{P^*_t}{Q_t Q^*_{st}} \right)^{\alpha_{sm}} = \frac{L_t^{\phi}}{C_t^{-\sigma}} \qquad \forall s \in S \tag{24}$$

$$\frac{\theta^*_s - 1}{\theta^*_s} Q_t \frac{P^*_{st}}{P^*_t} (X^*_{st})^{\frac{1}{\theta^*_s}} = \left( Z_{s,t} \left( \frac{\alpha_{sm}}{\alpha_{sl}} \frac{L_t^{\phi}}{C_t^{-\sigma}} \frac{P^*_t}{Q_t Q^*_{st}} \right)^{\alpha_{sm}} L_{st} - C_{st} \right)^{\frac{1}{\theta^*_s}} C_t \frac{\psi_s}{C_{st}} \qquad \forall s \in S \tag{25}$$

$$C_t = \xi C^*_t Q_t^{\frac{1}{\sigma}}, \tag{26}$$

and

$$D_0 = E_0 \sum_{t=0}^{\infty} \left[ \mathcal{M}^*_{0,t} P^*_t \sum_{s\in S} \left( \left( Z_{s,t} \left( \frac{\alpha_{sm}}{\alpha_{sl}} \frac{L_t^{\phi}}{C_t^{-\sigma}} \frac{P^*_t}{Q_t Q^*_{st}} \right)^{\alpha_{sm}} L_{st} - C_{st} \right)^{\frac{\theta^*_s - 1}{\theta^*_s}} X_{st}^{*\frac{1}{\theta^*_s}} \frac{P^*_{st}}{P^*_t} - \frac{\alpha_{sm}}{\alpha_{sl}} \frac{L_t^{\phi}}{C_t^{-\sigma}} \frac{L_{st}}{Q_t} \right) \right]. \tag{27}$$

To compare this with the flexible price allocation, I define the flexible price allocation as the solution to equations (1), (2), (4)-(6), (8), (13), (14), (15)-(18) under $\lambda_s = 0$ for all $s \in S$, and (19)-(23), and the household's budget constraint. Appendix B.2 shows that the equilibrium is characterized by the following:

$$C_t \frac{\psi_s}{C_{st}} \alpha_{sl} Z_{s,t} \left( \frac{\alpha_{sm}}{\alpha_{sl}} \frac{L_t^\phi}{C_t^{-\sigma}} \frac{P_t^*}{Q_t Q_{st}^*} \right)^{\alpha_{sm}} = \chi_s^{-1} \frac{L_t^\phi}{C_t^{-\sigma}} \qquad \forall s \in S \tag{28}$$

$$\frac{\theta_s^* - 1}{\theta_s^*} Q_t \frac{P_{st}^*}{P_t^*} \left(X_{st}^*\right)^{\frac{1}{\theta_s^*}} = \nu_s^{-1} \left( Z_{s,t} \left( \frac{\alpha_{sm}}{\alpha_{sl}} \frac{L_t^\phi}{C_t^{-\sigma}} \frac{P_t^*}{Q_t Q_{st}^*} \right)^{\alpha_{sm}} L_{st} - C_{st} \right)^{\frac{1}{\theta_s^*}} C_t \frac{\psi_s}{C_{st}} \qquad \forall s \in S \tag{29}$$

$$C_t = \xi C_t^* Q_t^{\frac{1}{\sigma}} \tag{30}$$

and

$$D_0 = E_0 \sum_{t=0}^{\infty} \left[ \mathcal{M}_{0,t}^* P_t^* \sum_{s \in S} \left( \left( Z_{s,t} \left( \frac{\alpha_{sm}}{\alpha_{sl}} \frac{L_t^\phi}{C_t^{-\sigma}} \frac{P_t^*}{Q_t Q_{st}^*} \right)^{\alpha_{sm}} L_{st} - C_{st} \right)^{\frac{\theta_s^*-1}{\theta_s^*}} X_{st}^{*\frac{1}{\theta_s^*}} \frac{P_{st}^*}{P_t^*} - \frac{\alpha_{sm}}{\alpha_{sl}} \frac{L_t^\phi}{C_t^{-\sigma} Q_t} L_{st} \right) \right], \tag{31}$$

where the real wedges $\chi_s, \nu_s$ are defined as

$$\chi_s = (1 - \tau_s) \left( \frac{\theta_s}{\theta_s - 1} \right)^{-1}, \ \nu_s = \frac{1 - \tau_s^X}{1 - \tau_s} \frac{\theta_s^*}{\theta_s^* - 1}.$$

We can see that the characterizations of allocations are equivalent except for the wedges $\chi_s$ and $\nu_s$. The wedge $\chi_s$ for all $s$ represents distortions coming from domestic monopolistic competition. The wedge $\nu_s$ for all $s$ represents distortions coming from the inability of the domestic firms to exert their monopolistic power in the international market, which I call the terms of trade externality.

Thus, the following lemma holds.

**Lemma 3.** *The flexible price allocation is efficient if and only if $\chi_s = \nu_s = 1$ for all $s \in S$. That is,*

$$1 - \tau_s = \frac{\theta_s}{\theta_s - 1}, \ 1 - \tau_s^X = (1 - \tau_s) \left( \frac{\theta_s^*}{\theta_s^* - 1} \right)^{-1} = \frac{\theta_s}{\theta_s - 1} \left( \frac{\theta_s^*}{\theta_s^* - 1} \right)^{-1}.$$

There are two types of inefficiency that the tax needs to address. To see this, note that even if the tax in each sector offsets the monopolistic markup in each sector by setting $1 - \tau_s = \theta_s / (\theta_s - 1)$ , inefficiency remains due to the difference

$$\frac{\theta_s}{\theta_s - 1} \left( \frac{\theta_s^*}{\theta_s^* - 1} \right)^{-1}$$

between $\theta_s$ and $\theta_s^*$. To achieve the efficient allocation, the tax needs to offset both internal distortion due to domestic monopolistic competition and external distortion due to (not utilizing) international monopolistic competition.

The external distortion arises when the elasticity of foreign demand is finite and hence $\theta_s^* / (\theta_s^* - 1) > 1$. In this case, the equilibrium consumption of export sector good is too low. The planner can improve welfare by exporting less while simultaneously improving the terms of trade. The market equilibrium cannot achieve this since each export sector takes the total demand for the exports as given, but the planner can strategically increase the sectoral price of exports as a whole to affect the terms of trade and foreign demand. To achieve this allocation in a decentralized manner, the fiscal authority needs to impose different tax rates depending on the destinations of goods.

In the following analysis, I assume such efficient tax rates to focus my analysis on monetary frictions. If I do not assume this efficient level of taxation, the monetary authority will have an incentive to use differential inflation rates across sectors to correct the distorted real allocation. If this force is added to the monetary trade-off that I analyze below, the analysis becomes too complicated. As the first step, I believe this simplification is beneficial in understanding the optimal price index.

## 3.2 Approximation of the Ramsey problem

This subsection derives the approximation to the Ramsey problem around the optimal steady state defined in Appendix B.3. I denote the log deviation from the steady state by the lowercase letter of the corresponding symbol of the variable. All domestic nominal variables are expressed relative to domestic CPI $P_t$. All international nominal variables are expressed in relative terms to foreign CPI $P_t^*$.

I show that when the steady state is efficient in the sense defined in the previous section, the second-order approximation of the welfare function, i.e., the objective function of the Ramsey problem, becomes purely quadratic without utilizing the second-order approximations of the pricing equations. Therefore, under regularity conditions, we can obtain an accurate first-order approximation to the solution of the non-linear Ramsey problem defined in Definition 1 by solving the approximated Ramsey problem that maximizes quadratically approximated welfare subject to linearly approximated constraints.

Note the difference between the optimal steady state and the efficient allocation. As mathematically defined in Appendix B.3, the optimal steady state is optimal in the second-best sense, where the monetary authority's problem takes sticky pricing mechanisms and market conditions as given. Therefore, the optimal steady state need not be an efficient allocation in the first-best sense. The appendix also shows that the optimal steady state can be characterized by the equations for flexible price allocation under constant exogenous variables and thus is efficient when the assumption of Lemma 3 is satisfied.

Denote the household's welfare by $\mathcal{W}$ and its steady state level by $\bar{\mathcal{W}}$. Define the vector of endogenous real variables as

$$v_t = [\boldsymbol{c}_t', \boldsymbol{x}_t']'$$

where

$$\boldsymbol{c}_t = [c_{1t}, ..., c_{St}]' \quad \text{and} \quad \boldsymbol{x}_t = [x_{1t}, ..., x_{St}]'$$

are the vectors of consumption and exports of all the sectors. Furthermore, define the vector of exogenous variables as

$$\xi_t = [c_t^*, \boldsymbol{x}_t^{*\prime}, \boldsymbol{p}_t^{*\prime}, \boldsymbol{q}_t^{*\prime}, \boldsymbol{z}_t']',$$

where

$$\boldsymbol{x}_t^* = [x_{1t}^*, ..., x_{St}^*]', \quad \boldsymbol{p}_t^* = [p_{1t}^*, ..., p_{St}^*]', \quad \boldsymbol{q}_t^* = [q_{1t}^*, ..., q_{St}^*]', \quad \text{and} \quad \boldsymbol{z}_t = [z_{1t}, ..., z_{St}]'$$

are the vectors of foreign demand for exports, international prices of exports, international prices of imports, and productivity shocks.

Before assuming the efficient tax rate, by using the market conditions except for the pricing equations, I show in Appendix B.5 that the approximated welfare can be written as

$$
\begin{aligned}
\frac{\mathcal{W}-\bar{\mathcal{W}}}{L^{1+\phi}} &= \sum_{t=0}^{\infty}\beta^t E_0 \boldsymbol{\phi}_l' d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(\boldsymbol{\phi}_c\right)\left(d\left(\boldsymbol{\chi}\right)^{-1}-I\right)\boldsymbol{c}_t \\
&+ \sum_{t=0}^{\infty}\beta^t E_0 \boldsymbol{\phi}_l' d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(\boldsymbol{\phi}_x\right)\left(d\left(\boldsymbol{\chi}\right)^{-1} d\left(\boldsymbol{\nu}\right)^{-1}-I\right)\boldsymbol{x}_t \\
&+ \frac{1}{2}\sum_{t=0}^{\infty}\beta^t E_0\left[\left(v_t-N\xi_t\right)'\Gamma_{v2}\left(v_t-N\xi_t\right)+\sum_{s\in S}\frac{\phi_{ls}}{\alpha_{sl}}\left[\phi_{sc}\frac{\theta_s}{\kappa_s}\pi_{s,t}^2+\phi_{sx}\frac{\theta_s}{\kappa_s}\left(\pi_{s,t}^X\right)^2\right]\right]+t.i.p.
\end{aligned}
$$

where $L$ is the steady-state level of aggregate labor supply, $\phi_{sc}=C_s/\left(C_s+X_s\right)$ is the steady-state consumption share of output in sector $s$, $\phi_{ls}=L_s/L$ is the steady-state labor usage share of sector $s$, $\phi_{sx}=1-\phi_{sc}$ is the steady-state export share of output and $d\left(\bullet\right)$ is the diagonal matrix of the vector inside the parentheses. The $2S$ by $4S+1$ matrix $N$ defines the natural levels $N\xi_t$ of the endogenous variables defined in the appendix.

The first two lines are linear in the endogenous variables, but when the steady state is efficient $\chi_s=\nu_s=1$ for all $s\in S$, all of the linear terms disappear. Therefore, under the efficient steady state, we can obtain a purely quadratic second-order approximation of welfare.

Appendix B.6 shows that under the efficient steady state, the natural levels of the endogenous variables coincide with the flexible price equilibrium denoted by $F\xi_t$ with a $2S$ by $4S+1$ matrix $F$. In the following, I denote the log deviation from the flexible price equilibrium by $\tilde{v}_t := v_t - F\xi_t$. Furthermore, from the following relationship obtained in Appendix B.4

$$
\psi_s=\frac{\chi_s^{-1}\phi_{sc}\frac{\phi_{ls}}{\alpha_{sl}}}{\sum_{s'\in S}\chi_{s'}^{-1}\phi_{sc'}\frac{\phi_{ls'}}{\alpha_{s'l}}},
$$

we can see that the coefficients of the inflation rates can be simplified to

$$
\sum_{s\in S}\frac{\phi_{ls}}{\alpha_{sl}}\left[\phi_{sc}\frac{\theta_s}{\kappa_s}\pi_{s,t}^2+\phi_{sx}\frac{\theta_s}{\kappa_s}\left(\pi_{s,t}^X\right)^2\right]=\left(\sum_{s'\in S}\phi_{sc'}\frac{\phi_{ls'}}{\alpha_{s'l}}\right)\sum_{s\in S}\frac{\theta_s}{\kappa_s}\psi_s\left[\pi_{s,t}^2+\frac{\phi_{sx}}{\phi_{sc}}\left(\pi_{s,t}^X\right)^2\right].
$$

Therefore, I obtain the following lemma.

**Lemma 4.** *If the steady state is efficient, approximated optimal monetary policy can be obtained by solving the linear-quadratic problem. Given initial conditions $v_{-1}$ and precommitment, the central bank chooses $\left\{\tilde{v}_t, \boldsymbol{\pi}_t, \boldsymbol{\pi}_t^X, \pi_t\right\}_{t=0}^{\infty}$ to minimize*

$$
\sum_{t=0}^{\infty}\beta^t E_0\left[\tilde{v}_t'\Gamma_{v2}\tilde{v}_t+\Gamma_\pi\sum_{s\in S}\frac{\theta_s}{\kappa_s}\psi_s\left[\pi_{s,t}^2+\frac{\phi_{sx}}{\phi_{sc}}\left(\pi_{s,t}^X\right)^2\right]\right]
$$

*subject to (1) the Phillips curves*

$$
d\left(\boldsymbol{\kappa}\right)^{-1}\left(\boldsymbol{\pi}_t-\beta E_t\left[\boldsymbol{\pi}_{t+1}\right]\right)=\gamma_v^P\tilde{v}_t \quad \text{and} \quad d\left(\boldsymbol{\kappa}\right)^{-1}\left(\boldsymbol{\pi}_t^X-\beta E_t\left[\boldsymbol{\pi}_{t+1}^X\right]\right)=\gamma_{Xv}^P\tilde{v}_t,
$$

*where $\kappa_s = (1-\lambda_s)(1-\beta\lambda_s)/\lambda_s$, and (2) the identities linking inflation rates and relative prices*

$$\boldsymbol{\pi}_t = \mathbf{1}_{S\times 1}\pi_t + \gamma_v^I(\tilde{v}_t - \tilde{v}_{t-1}) + \epsilon_t^I - \epsilon_{t-1}^I \text{ and } \boldsymbol{\pi}_t^X = \mathbf{1}_{S\times 1}\pi_t + \gamma_{vX}^I(\tilde{v}_t - \tilde{v}_{t-1}) + \epsilon_t^{IX} - \epsilon_{t-1}^{IX}.$$

*Proof.* See Appendix B.7. □

The coefficient matrices $\Gamma_{v2}$, $\gamma_v^P$, $\gamma_{vX}^P$, $\gamma_v^I$ and $\gamma_{vX}^I$ , the scalar $\Gamma_\pi$ and the residuals $\epsilon_t^I$ and $\epsilon_t^{IX}$are given in Appendix B.7. The choice variables are the vector of consumption of each sector $\boldsymbol{c}_t$ and the vector of exports from each sector $\boldsymbol{x}_t$ contained in the vector of endogenous variables $v_t$, the vector of inflation rates

$$\boldsymbol{\pi}_t = [\pi_{1t}, ..., \pi_{St}]', \ \boldsymbol{\pi}_t^X = \left[\pi_{1t}^X, ..., \pi_{St}^X\right]'$$

and CPI inflation $\pi_t$. The reason for having CPI inflation here is that nominal variables are normalized by CPI inflation. One can alternatively write the equations with different normalization and still obtain the same result for the optimal price index.

As is usual in closed economy analysis, we have two parts in the objective function. The first part is the quadratic terms in the gaps in real variables from their respective natural levels. The second part is the nominal part representing the cost of volatile inflation.

The nominal friction is larger when the sector uses more labor, the price is sticky, or the elasticity of substitution is high. This is intuitive because if the inflation rate is volatile in a sector, the price dispersion of the sector increases. This means that to produce a certain effective output in the sector, the sector requires more labor input and imported materials, causing disutility for the household through more labor or a tighter international budget. The overall effect will be larger if the sector uses more labor at the steady state. Inflation volatility leads to higher price dispersion when the price is stickier. Given the same distribution of individual prices within a sector, the degrees of price dispersion $\Delta_{st}, \Delta_{st}^X$ become higher if the elasticity of substitution $\theta_s$ is higher.

In the constraints, there are in total $2S$ Phillips curves for domestic prices and export prices in each sector. The last two equations in the constraints are identities linking sectoral inflation rates $\boldsymbol{\pi}_t, \boldsymbol{\pi}_t^X$ and CPI inflation $\pi_t$. This means that there is only one degree of freedom left in this problem. Although there are different inflation rates for different sectors, they cannot be freely chosen since relative inflation rate between two sectors determines the evolution of the relative price of the two sectors.

## 3.3 Ramsey price index

This subsection states the main result of this paper. If we define a price index using the coefficients on the inflation rates in the loss function derived in the previous subsection, the price index stays constant in the long-run expectation under the optimal monetary policy. This implies that if the central bank does not stabilize this price index in the long-run, its policy is necessarily sub-optimal. Specifically, Appendix B.8 shows the following.

**Theorem 5.** *Define the price index as*

$$\log \mathbb{P}_t = \Phi^{-1}\sum_{s\in S}\psi_s\frac{\theta_s}{\kappa_s}\left(\log P_{st} + \frac{\phi_{sx}}{\phi_{sc}}\log P_{st}^X\right)$$

*with*

$$\Phi = \sum_{s \in S} \psi_s \frac{\theta_s}{\kappa_s} \left(1 + \frac{\phi_{sx}}{\phi_{sc}}\right).$$

*Then, under the solution to the Ramsey problem,*

$$\lim_{T \to \infty} E_t \log \mathbb{P}_T = \Phi^{-1} \overline{\log \mathbb{P}}.$$

I call this price index $\mathbb{P}_t$ the RPI since its stabilization is desirable as the solution to the Ramsey problem. The scalar $\Phi$ is used to normalize the coefficients to sum to one. This theorem states that the long-run stabilization of the RPI can be obtained as a necessary condition of the solution of the Ramsey problem. The theorem motivates the central bank's policy that stabilizes the inflation rate measured in this index since if this price index is not stabilized in the long run under some policy, the policy must be sub-optimal.

The converse is not necessarily true. That is, complete stabilization of this price index does not necessarily guarantee that the economy follows the optimal path consistent with the first-order conditions. Although it is generally possible to derive the if-and-only-if condition using the method of Giannoni and Woodford [2010], the condition is generally complicated. To keep my discussion simple and intuitive, I propose the use of a simple policy rule that always stabilizes the RPI. The welfare analysis in Section 4 shows that the welfare loss from simple RPI stabilization policy is negligible compared to the optimal monetary policy and that it performs better than the stabilization of headline CPI, core CPI, and PPI.

The RPI is a weighted sum of prices in different sectors, where the weights depend on consumption share $\psi_s$, the elasticity of substitution $\theta_s$, the Phillips curve slope $\kappa_s$ that contains the information of the price stickiness $\lambda_s$ and the trade pattern $\phi_{sx}/\phi_{sc}$.

The weight reflects the trade-off that the monetary authority faces. As the derivation indicates, the weight takes the form of the coefficients on inflation rates in the loss function of the Ramsey problem representing the cost of inflation in different sectors. If the volatility of the inflation rate in a sector is relatively more costly to welfare than that in other sectors, the RPI assigns higher weight to the former sector.

Note that this price index will remain constant even if there is a unit-root process in the exogenous variables that may result in a permanent change in the natural levels of endogenous variables. This fact should be noted since if all exogenous variables are stationary, price levels under any price index will eventually coincide after all transitory shocks die out.

### 3.3.1 Comparison with CPI and PPI

To understand the relationship between the RPI and the conventional price indices, let us consider the weight on sector $s$ under $\log P_s = \log P_s^X$. Recalling that $\phi_{sc} + \phi_{sx} = 1$, the weight on the price in sector $s$ becomes

$$\psi_s \frac{\theta_s}{\kappa_s} \left(1 + \frac{\phi_{sx}}{\phi_{sc}}\right) = \frac{\theta_s}{\kappa_s} \underbrace{\overbrace{\psi_s}^{CPI} \frac{1}{\phi_{sc}}}_{PPI}.$$

From this expression, we can see that weighting under the RPI can be seen as that under PPI multiplied by the sensitivity of the wedge to inflation $\theta_s/\kappa_s$. The PPI weight is relevant because the cost of inflation appears as the wedge in production; see equation (20). Therefore, the relevant size of the sector is the production size rather than consumption size.

However, the quantitative result in the next section shows that the sensitivity of the wedge to inflation $\theta_s/\kappa_s$ is important in the sense that PPI targeting sometimes performs worse than CPI targeting. The reason for the inclusion of this additional factor is that a given inflation volatility causes different wedge sizes depending on price stickiness, summarized by $\kappa_s$, and the elasticity of substitution, captured by $\theta_s$.

Compared to the CPI weight, $\psi_s$, the PPI weight is higher for exporting sectors. This is because when some of the output is exported, the consumption weight on the sector is smaller than the optimal weight. In such a case, we can obtain the correct size of the sector by inflating the consumption weight $\psi_s$ by the output-to-consumption ratio $1/\phi_{sc}$.

We can also obtain the price index derived in Woodford [2010] as a special case by assuming no trade $\phi_{sc} = 1$ and a homogeneous elasticity of substitution $\theta_s = \theta$. In this special case, the weight assigned to sector $s$ is[6] $\psi_s/\kappa_s$.

The previous literature has argued for core inflation stabilization based on the observation that the non-core sectors have higher degrees of flexibility or higher values of $\kappa_s$, resulting in disproportionately smaller weights on those sectors. The RPI adjusts for the elasticity of substitution $\theta_s$ and trade $1/\phi_{sc}$. The former has the effect of placing a higher weight on sectors with higher substitutability within the sector. This is important since some non-core sectors do have higher values of the elasticity of substitution. The latter has the effect of placing a higher weight on export sectors. This may shift the optimal weight away from the core weight and closer to the headline weight for commodity exporting countries.

### 3.3.2 Role of international commodity prices

Another lesson that we can learn from the formula for RPI is that international commodity prices $P^*_{st}, Q^*_{st}$ do not appear directly in the index. That is, the formula for RPI in Theorem 5 is a weighted sum of prices set by domestic firms $P_{st}$ and $P^X_{st}$. Even if those prices are influenced by international prices, the formula does not adjust for or offset the influence of external factors.

Note that this is despite the fact that I naturally model the effect of exogenous international prices. As in the pricing equations (13)-(18), the international price of inputs $Q^*_{st}$ affect the firms' pricing behavior through their marginal costs. As in the export demand equation (23), prices of international competitors $P^*_{st}$ affect export demand. The former has a first-order impact on sectoral prices, and the latter has a first-order impact on the trade balance and a second-order impact on sectoral prices.

We can observe from the formula in Theorem 5 that these international prices affect the optimal price index if and only if they affect the output prices of domestic sectors. This is because volatile inflation causes efficiency loss in production regardless of the cause of the volatility, and thus, we do not need to adjust the formula for the price index depending on whether such volatility comes from international prices. In other words, output prices in the formula are sufficient statistics in the measure of the most welfare-relevant inflation rate.

[6] This is not exactly the same as the expression in Woodford (2010) since I am simplifying the analysis in one dimension, namely, heterogeneity in the labor. This will affect the expression for the $\kappa_s$ reflecting the increasing disutility from uneven labor supply.

As an implication, although we may tend to think that central banks are not responsible for inflation volatility caused by international price movements, a central bank should be concerned about volatility as long as it affects the RPI. To understand this point, note that although international prices are exogenous, domestic prices can be controlled via changes in the exchange rate. Imagine an economy where all the domestic prices of different sectors are proportional to the international prices in those sectors. The ratio between the vector of international prices and that of domestic prices is the exchange rate. If the central bank selects one domestic sector, it is possible to stabilize the domestic price of that sector by adjusting the exchange rate to offset international price movements. Of course, this operation affects all other sectors, so the central bank faces a trade-off between stabilizing one sector and stabilizing another. The RPI indicates how to balance this trade-off.

# 4 Quantitative Results

This section calibrates the model to data on 40 countries with 35 sectors. The purpose of the calibration is twofold: first, to understand the quantitative difference between the optimal price index and conventional price indices and, second, to obtain some insights into the implementation of the optimal monetary policy. That is, as noted above, the long-run stabilization of the optimal price index is insufficient to guarantee that the economy follows the optimal path. Therefore, the performance of the simple policy rule that completely stabilizes the optimal price index would be of interest. I calculate the welfare loss from stabilizing the optimal price index and sub-optimal price indices.

## 4.1 Welfare evaluation

I compare the welfare under the solution to the Ramsey problem, i.e., the optimal policy with those under four simple stabilization policies for the RPI, headline CPI, core CPI, and PPI. The equilibrium dynamics can be obtained by solving for the bounded solution of the set of constraints combined with one of the following monetary policy alternatives.

1. Optimal monetary policy characterized by the first-order conditions (35).

2. RPI stabilization[7]

$$\sum_{s \in S} \frac{\theta_s}{\kappa_s} \psi_s \left[ \pi_{s,t} + \frac{\phi_{sx}}{\phi_{sc}} \pi^X_{s,t} \right] = 0$$

3. Headline CPI stabilization.

$$\pi_t = 0$$

4. Core CPI stabilization. Denoting the set of core sectors by $Core \subset S$,

$$\sum_{s \in Core} \psi_s \pi_{st} = 0$$

[7]In case $\phi_{sc} = 0$, I use the original expression of the weight $\sum_{s \in S} \frac{\phi_{ls}}{\alpha_{sl}} \frac{\theta_s}{\kappa_s} \left[ \phi_{sc} \pi_{s,t} + \phi_{sx} \pi^X_{s,t} \right]$.

Table 1: Parameters common across countries and sectors

| | Parameter | Value | Note |
|---|---|---|---|
| $\beta$ | Discount rate | $0.97^{\frac{1}{12}}$ | 3% annual rate |
| $\sigma$ | Inverse intertemporal elasticity of substitution | 2 | e.g. Arellano [2008] |
| $\phi$ | Inverse Frisch elasticity of labor supply | 0.47 | e.g. De Paoli [2009] |

5. PPI stabilization. Denoting the steady state output by $Y_s, Y_s^X$ for all $s \in S$,

$$\sum_{s \in S} \left( Y_s \pi_{st} + Y_s^X \pi_{st}^X \right) = 0$$

I evaluate the welfare

$$\mathcal{W} - \bar{\mathcal{W}} = \frac{1}{2} L^{1+\phi} \sum_{t=0}^{\infty} \beta^t E_0 \left[ (v_t - N\xi_t)' \Gamma_{v2} (v_t - N\xi_t) + \sum_{s \in S} \frac{\phi_{ls}}{\alpha_{sl}} \left[ \phi_{sc} \frac{\theta_s}{\kappa_s} \pi_{s,t}^2 + \phi_{sx} \frac{\theta_s}{\kappa_s} \left( \pi_{s,t}^X \right)^2 \right] \right] + t.i.p.$$

under each of the solutions and report the welfare loss compared to the optimal monetary policy.

## 4.2 Data

To evaluate the welfare loss described in the previous subsection, I need to obtain parameter values, some steady-state variables a description of the exogenous processes. I consider one period to be one month in this section. Parameters common across all countries and sectors, summarized in Table 1, are the discount factor $\beta = 0.97^{\frac{1}{12}}$, to match the 3% annual discount rate, the inverse of the elasticity of intertemporal substitution $\sigma = 2$, which is the standard value in the literature, and the inverse of the Frisch elasticity of labor supply $\phi = 0.47$, following De Paoli (2009)De Paoli [2009].

I allow for sectoral heterogeneity in the elasticity of substitution $\theta_s$ and price stickiness $\lambda_s$ . For the stickiness parameters, I use the estimates of Nakamura and Steinsson [2008]. For the elasticity of substitution, I use the estimates of Broda and Weinstein [2006]. I follow the categorization of 35 industrial sectors in the World Input-Output Database (WIOD)[8]. Appendix C.2 shows the concordance of the categories across these data sources. The parameter values are summarized in Table 2. In the analysis below, these stickiness parameters and elasticity parameters are assumed to be common across countries.

Since the definition of the "core" index varies across countries, I define the set of core sectors $Core \subset S$ as non-commodity sectors for the purposes of cross-country comparison. Table 2 also reports whether a sector is the core sector.

I use country-specific values for $\boldsymbol{\psi}, \boldsymbol{\alpha}_m, \boldsymbol{\alpha}_l, \boldsymbol{\phi}_c, \boldsymbol{\phi}_x$ and $\boldsymbol{\phi}_l$. These are constructed for 40

[8]See Timmer et al. [2015].

Table 2: Sector-specific parameters common across all countries

| Sector | WIOD | $\theta_s$ | $\lambda_s$ | $\frac{\theta_s}{\kappa_s}$ | Core |
|---|---|---|---|---|---|
| 1 | Agriculture, Hunting, Forestry and Fishing | 9.83 | .125 | 2 | 1 |
| 2 | Mining and Quarrying | 5.53 | .961 | 3289 | 1 |
| 3 | Food, Beverages and Tobacco | 6.35 | .737 | 67 | 0 |
| 4 | Textiles and Textile Products | 3.91 | .977 | 6519 | 1 |
| 5 | Leather, Leather and Footwear | 3.69 | .962 | 2310 | 1 |
| 6 | Wood and Products of Wood and Cork | 4.01 | .987 | 19639 | 1 |
| 7 | Pulp, Paper, Paper , Printing and Publishing | 5.05 | .956 | 2364 | 1 |
| 8 | Coke, Refined Petroleum and Nuclear Fuel | 5.75 | .513 | 12 | 0 |
| 9 | Chemicals and Chemical Products | 5.25 | .939 | 1275 | 1 |
| 10 | Rubber and Plastics | 4.8 | .968 | 4214 | 1 |
| 11 | Other Non-Metallic Mineral | 3.04 | .959 | 1637 | 1 |
| 12 | Basic Metals and Fabricated Metal | 7.43 | .962 | 4651 | 1 |
| 13 | Machinery, Nec | 8.99 | .963 | 5932 | 1 |
| 14 | Electrical and Optical Equipment | 4.79 | .963 | 3161 | 1 |
| 15 | Transport Equipment | 13.41 | .727 | 130 | 1 |
| 16 | Manufacturing, Nec; Recycling | 2.75 | .835 | 83 | 1 |
| 17 | Electricity, Gas and Water Supply | 2.59 | .513 | 6 | 0 |
| 18 | Construction | 2.59 | .939 | 629 | 1 |
| 19 | Sale, Maintenance and Repair of Motor Vehicles and Motorcycles; Retail Sale of Fuel | 2.59 | .531 | 6 | 0 |
| 20 | Wholesale Trade and Commission Trade, Except of Motor Vehicles and Motorcycles | 2.59 | .939 | 629 | 1 |
| 21 | Retail Trade, Except of Motor Vehicles and Motorcycles; Repair of Household Goods | 2.59 | .939 | 629 | 1 |
| 22 | Hotels and Restaurants | 2.59 | .939 | 629 | 1 |
| 23 | Inland Transport | 2.59 | .583 | 9 | 1 |
| 24 | Water Transport | 2.59 | .583 | 9 | 1 |
| 25 | Air Transport | 2.59 | .583 | 9 | 1 |
| 26 | Other Supporting and Auxiliary Transport Activities; Activities of Travel Agencies | 2.59 | .583 | 9 | 1 |
| 27 | Post and Telecommunications | 2.59 | .939 | 629 | 1 |
| 28 | Financial Intermediation | 2.59 | .939 | 629 | 1 |
| 29 | Real Estate Activities | 2.59 | .939 | 629 | 1 |
| 30 | Renting of MandEq and Other Business Activities | 2.33 | .939 | 566 | 1 |
| 31 | Public Admin and Defence; Compulsory Social Security | 2.59 | .939 | 629 | 1 |
| 32 | Education | 2.59 | .939 | 629 | 1 |
| 33 | Health and Social Work | 2.59 | .939 | 629 | 1 |
| 34 | Other Community, Social and Personal Services | 2.85 | .939 | 692 | 1 |
| 35 | Private Households with Employed Persons | 2.59 | .939 | 629 | 1 |

countries in the 2013 release of World Input-Output Database as follows[9]. I use the year 2000 to align with the periods covered in other estimates (Nakamura and Steinsson: 1998-2005, Broda and Weinstein 1990-2001) and the 2013 release for the sake of matching with Rauch's classification.

For a given country, the domestic part of its input-output table is taken from the WIOD and the imports and exports are calculated by summing all the foreign entries for the country. As consumption $\{P_s C_s\}_{s\in S}$, I use the sum of gross fixed capital formation (WIOD column c41) and final consumption by households (c37), non-profit organizations serving households (c38), and government for each sector (c39). The consumption expenditure share $\boldsymbol{\psi}$ is calculated as the share of each sector over aggregate domestic consumption.

As the payment to labor $\{WL_s\}_{s\in S}$, I use value added (WIOD row r64). The labor usage share $\boldsymbol{\phi}_l$ is calculated as the share of each sector over the aggregate value added of all the sectors in the country.

Since I abstract from the input-output linkages in my theoretical analysis, I need to obtain the values of $\boldsymbol{\alpha}_m, \boldsymbol{\alpha}_l, \boldsymbol{\phi}_c, \boldsymbol{\phi}_x$ that correspond to the economy without input-output linkages. To do so, I adjust the raw input shares and usage shares using the input-output matrix. The adjustment described in Appendix C.3 counts all indirect usages of labor and imported goods in calculating $\boldsymbol{\alpha}_m, \boldsymbol{\alpha}_l$. In calculating $\boldsymbol{\phi}_c, \boldsymbol{\phi}_x$, all indirect consumption and exports are counted. In this way, I can obtain the property $\boldsymbol{\alpha}_m + \boldsymbol{\alpha}_l = \mathbf{1}_{S\times 1}$ assumed in the analysis and the property $\boldsymbol{\phi}_c + \boldsymbol{\phi}_x = 1$ that needs to hold by definition.

Finally, the dynamics of the exogenous variables are assumed to be described as a vector auto-regressive process with one lag (VAR(1)). I obtain the coefficients and the variance-covariance matrix of the error terms by fitting the following monthly processes to the VAR(1) model. The sample period is from June 2009 to August 2017.

I use the logarithm of US consumption as world consumption $c_t^*$, US imports as an approximation of world demand $\boldsymbol{x}_t^*$, and US export price indices as an approximation of the prices of international competitors $\boldsymbol{p}_t^*$. The monthly series are accessed through CEIC[10], and the data sources are summarized in Table 3 for export demand $\boldsymbol{x}_t^*$ and in Table 4 for export prices $\boldsymbol{p}_t^*$. For $c_t^*$, I use seasonally adjusted series of personal consumption expenditure (PCE) in 2012 prices from Bureau of Economic Analysis. The standard deviation in the sample is 0.94%.

For import prices $\boldsymbol{q}_t^*$, I combine export price indices using country-specific compositions of imports to sectors. That is, I use the World Input-Output table to calculate how much sector $s$ of a given country imports goods and services from sector $s'$ of all other countries. I denote the share of imports from sector $s'$ over total imports to sector $s$ by $\tilde{\alpha}_{ss'}$. I then use the weighted sum of the log prices of all source sectors $s'$ as the import price index $q_{st}^* = \sum_{s'\in S} \tilde{\alpha}_{s's} p_{s't}^*$. I assume that productivity $\boldsymbol{z}_t$ is constant to focus on observable shocks.

[9]The countries included are Australia, Austria, Belgium, Bulgaria, Brazil, Canada, China, Cyprus, Czech Republic, Germany, Denmark, Spain, Estonia, Finland, France, the United Kingdom, Greece, Hungary, India, Indonesia, Ireland, Italy, Japan, Korea, Lithuania, Luxembourg, Latvia, Mexico, Malta, the Netherlands, Poland, Portugal, Romania, Russia, Slovakia, Slovenia, Sweden, Turkey, Taiwan, and the U.S.A.

[10]CEIC is a proprietary database, which can be accessed here: https://insights.ceicdata.com.

Table 3: Data source

| x* | WIOD | Std (%) | Series Name | Source |
|---|---|---|---|---|
| 1 | Agriculture, Hunting, Forestry and Fishing | 12.7 | Imports: 1-Digit: Food and Live Animals | US Census Bureau |
| 2 | Mining and Quarrying | 9.5 | Import Value: SITC: Customs, Aggregate under Metal and Mining Sector | US Census Bureau |
| 3 | Food, Beverages and Tobacco | 10.2 | Imports: 1-Digit: Beverages and Tobacco | US Census Bureau |
| 4 | Textiles and Textile Products | 10.6 | Imports: CIF: 2-Digit: Textile Fibers and Their Wastes | US Census Bureau |
| 5 | Leather, Leather and Footwear | 7 | Imports: 1-Digit: Manufactured Goods Classified Chiefly by Material | US Census Bureau |
| 6 | Wood and Products of Wood and Cork | 9.9 | Imports: 2-Digit: Cork and Wood | US Census Bureau |
| 7 | Pulp, Paper, Paper , Printing and Publishing | 6.5 | Imports: 2-Digit: Paper, Paperboard and Pulp | US Census Bureau |
| 8 | Coke, Refined Petroleum and Nuclear Fuel | 9.9 | Imports: 2-Digit: Petroleum, Petroleum Products | US Census Bureau |
| 9 | Chemicals and Chemical Products | 6.8 | Imports: 1-Digit: Chemicals and Related Products, nes | US Census Bureau |
| 10 | Rubber and Plastics | 11.2 | Imports: 2-Digit: Rubber Manufactures | US Census Bureau |
| 11 | Other Non-Metallic Mineral | 8.4 | Imports: NAICS: Mfg: Non Metallic Mineral | US Census Bureau |
| 12 | Basic Metals and Fabricated Metal | 13.2 | Imports: 2-Digit: Metalliferous Ores and Metal Scrap | US Census Bureau |
| 13 | Machinery, Nec | 8.1 | Imports: 1-Digit: Machinery and Transport Equipment | US Census Bureau |
| 14 | Electrical and Optical Equipment | 8.5 | Imports: 2-Digit: Electrical Machinery, Apparatus and Appliances, nes | US Census Bureau |
| 15 | Transport Equipment | 10.3 | Imports: 2-Digit: Road Vehicles | US Census Bureau |
| 16 | Manufacturing, Nec; Recycling | 9 | Imports: 1-Digit: Miscellaneous Manufactured Articles | US Census Bureau |
| 17 | Electricity, Gas and Water Supply | 22.4 | Imports: 2-Digit: Electric Current | US Census Bureau |
| 18 | Construction | 1.7 | Imports: sa: Service | US Census Bureau |
| 19 | Sale, Maintenance and Repair of Motor Vehicles and Motorcycles; Retail Sale of Fuel | 10.5 | Imports: 2-Digit: Road Vehicles | US Census Bureau |
| 20 | Wholesale Trade and Commission Trade, Except of Motor Vehicles and Motorcycles | 7.8 | Merchant Wholesalers Sales: Total | US Census Bureau |
| 21 | Retail Trade, Except of Motor Vehicles and Motorcycles; Repair of Household Goods | 6.8 | Retail Sales and Food Services: ex Motor Vehicle and Parts | US Census Bureau |
| 22 | Hotels and Restaurants | 5.4 | Retail Sales: FS: ow: Full Service Restaurants | US Census Bureau |
| 23 | Inland Transport | 5.8 | PCE: saar: SE: HCE: TR: PT: Ground Transportation (GT) | Bureau of Economic Analysis |
| 24 | Water Transport | 2.8 | Imports: sa: Service | US Census Bureau |
| 25 | Air Transport | 12.1 | PCE: saar: SE: HCE: TR: PT: Air Transportation | Bureau of Economic Analysis |
| 26 | Other Supporting and Auxiliary Transport Activities; Activities of Travel Agencies | 5.4 | Avg Weekly Earnings: PB: Travel Agency | Bureau of Labor Statistics |
| 27 | Post and Telecommunications | 4.6 | PCE: saar: SE: HCE: Other: CO: Postal and Delivery Services (PDS) | Bureau of Economic Analysis |
| 28 | Financial Intermediation | 2.7 | Avg Weekly Earnings: FA: Credit Intermediation and Rel Activities | Bureau of Labor Statistics |
| 29 | Real Estate Activities | 2.6 | Avg Weekly Earnings: FA: Real Estate | Bureau of Labor Statistics |
| 30 | Renting of MandEq and Other Business Activities | 6 | Avg Weekly Earnings: FA: Machinery and Equip Rental and Leasing | Bureau of Labor Statistics |
| 31 | Public Admin and Defence; Compulsory Social Security | 2.2 | Imports: sa: Service | US Census Bureau |
| 32 | Education | 1.7 | Imports: sa: Service | US Census Bureau |
| 33 | Health and Social Work | 1.4 | Imports: sa: Service | US Census Bureau |
| 34 | Other Community, Social and Personal Services | 2.4 | Avg Weekly Earnings: OS: Personal Care Services | Bureau of Labor Statistics |
| 35 | Private Households with Employed Persons | 1.5 | Imports: sa: Service | US Census Bureau |

## Table 4: Data source

| p* | WIOD | Std (%) | Series Name | Source |
|---|---|---|---|---|
| 1 | Agriculture, Hunting, Forestry and Fishing | 15.2 | Export Price Index: Agriculture and Livestock Products (ALP) | Bureau of Labor Statistics |
| 2 | Mining and Quarrying | 14.2 | Export Price Index: Oil, Gas, Meneral and Ores: Mineral and Ores | Bureau of Labor Statistics |
| 3 | Food, Beverages and Tobacco | 6.3 | Export Price Index: Beverages and Tobacco Products | Bureau of Labor Statistics |
| 4 | Textiles and Textile Products | 7.8 | Export Price Index: Textile and Textile Articles (TA) | Bureau of Labor Statistics |
| 5 | Leather, Leather and Footwear | 9 | PPI: Hides, Skins, Leather and Products | Bureau of Labor Statistics |
| 6 | Wood and Products of Wood and Cork | 3.1 | (DC)Export Price Index: Wood Products | Bureau of Labor Statistics |
| 7 | Pulp, Paper, Paper , Printing and Publishing | 3.2 | Export Price Index: Paper | Bureau of Labor Statistics |
| 8 | Coke, Refined Petroleum and Nuclear Fuel | 26.1 | Export Price Index: Petroleum and Coal Products | Bureau of Labor Statistics |
| 9 | Chemicals and Chemical Products | 5.7 | Export Price Index: Chemicals | Bureau of Labor Statistics |
| 10 | Rubber and Plastics | 3.4 | Export Price Index: Plastics and Rubber Products (PRP) | Bureau of Labor Statistics |
| 11 | Other Non-Metallic Mineral | .8 | Export Price Index: Nonmatalic Mineral Products | Bureau of Labor Statistics |
| 12 | Basic Metals and Fabricated Metal | 9.8 | Export Price Index: Primary Metals (PM) | Bureau of Labor Statistics |
| 13 | Machinery, Nec | 1.1 | Export Price Index: Machinery (MA) | Bureau of Labor Statistics |
| 14 | Electrical and Optical Equipment | .6 | Export Price Index: Computer and Electronics Products (CEP) | Bureau of Labor Statistics |
| 15 | Transport Equipment | .6 | Export Price Index: Transportation Equipment | Bureau of Labor Statistics |
| 16 | Manufacturing, Nec; Recycling | .5 | Export Price Index: Miscellaneous Manufactured Articles (MM) | Bureau of Labor Statistics |
| 17 | Electricity, Gas and Water Supply | 1.3 | CPI U: Services: Utilities and Public Transportation | Bureau of Labor Statistics |
| 18 | Construction | 1.3 | PPI: ME: Construction | Bureau of Labor Statistics |
| 19 | Sale, Maintenance and Repair of Motor Vehicles and Motorcycles; Retail Sale of Fuel | .2 | CPI U: Transport: Private: MV Maintenance and Repair (MR) | Bureau of Labor Statistics |
| 20 | Wholesale Trade and Commission Trade, Except of Motor Vehicles and Motorcycles | 1 | PPI: Wholesale Trade Services (WTS) | Bureau of Labor Statistics |
| 21 | Retail Trade, Except of Motor Vehicles and Motorcycles; Repair of Household Goods | 1 | CPI U: Housing: HFO: HO: Repair of Household Items | Bureau of Labor Statistics |
| 22 | Hotels and Restaurants | 3.6 | PPI: Accommodation Services: Travel Accommodation | Bureau of Labor Statistics |
| 23 | Inland Transport | 3.6 | PPI: Travel Arrangement Services: Vehicle Rentals and Lodging | Bureau of Labor Statistics |
| 24 | Water Transport | 2.7 | PPI: Travel Arrangement Services: Cruises and Tours | Bureau of Labor Statistics |
| 25 | Air Transport | 11.1 | Export Price Index: Air Passenger Fares | Bureau of Labor Statistics |
| 26 | Other Supporting and Auxiliary Transport Activities; Activities of Travel Agencies | 2.1 | PPI: Travel Arrangement Services: Others | Bureau of Labor Statistics |
| 27 | Post and Telecommunications | 1.4 | PPI: ME: General: Scales and Balances: Retail,Commercial,Hseholdand Mail | Bureau of Labor Statistics |
| 28 | Financial Intermediation | 2.1 | PPI: Credit Intermediation Services (CIS) | Bureau of Labor Statistics |
| 29 | Real Estate Activities | 2 | PPI: Real Estate Services | Bureau of Labor Statistics |
| 30 | Renting of MandEq and Other Business Activities | 3.6 | PPI: Rental and Leasing of Goods | Bureau of Labor Statistics |
| 31 | Public Admin and Defence; Compulsory Social Security | 1.1 | PPI: Selected Security Services | Bureau of Labor Statistics |
| 32 | Education | 1 | PPI: Educational Services | Bureau of Labor Statistics |
| 33 | Health and Social Work | .6 | CPI U: Medical Care: Services | Bureau of Labor Statistics |
| 34 | Other Community, Social and Personal Services | .7 | CPI U: GS: PC: Personal Care Services | Bureau of Labor Statistics |
| 35 | Private Households with Employed Persons | .2 | PCE: PI: sa: Services (SE) | Bureau of Economic Analysis |

### 4.3 Welfare results

Table 5 shows the welfare loss from simple monetary policy rules (i.e., monetary policies 2-5 in Subsection 4.1) compared with the optimal monetary policy. The units for these values is 0.01% of steady-state consumption.

As a benchmark, notice that the welfare loss from the stabilization of conventional price indices reported in Table 5 is on the order of 0.01% of the steady-state consumption. This is small as a percentage of consumption, but it is typical to obtain such numbers in the standard NK environment. For example, Gali and Monacelli [2005] report 0.0166% for their benchmark case.

The first finding from the welfare calibration is that most of the welfare loss can be eliminated by switching from stabilizing conventional price indices to the RPI. Comparing the second column, labeled Ramsey, with any of the third to the fourth columns in Table 5, the welfare loss in terms of consumption decreases to less than one-hundredth of the loss from targeting conventional indices, on average across countries. In other words, mere stabilization of RPI performs as well as the solution to the Ramsey problem.

The second finding shown in Table 5 is that, while RPI is always the best, the ranking of the stabilization of other indices varies across countries. This implies that we should not conclude that PPI is superior to CPI just because the analytical expression for the RPI can be interpreted as PPI plus an adjustment. For example, the worst index to target for the U.S., China, and Japan is PPI, core CPI, and headline CPI, respectively. In other words, the adjustment is large enough to make PPI stabilization less desirable than CPI stabilization for some countries, depending on the trade pattern.

## 5 Conclusion

In this paper, I solve a central bank's Ramsey problem and derive the Ramsey price index for small open economies to stabilize. Due to the openness of my model, the index depends on the export share of output in each sector in addition to those parameters that have been studied in closed economy models such as the consumption share, price stickiness and the elasticity of substitution.

By calibrating the formula to 40 countries, I find that RPI stabilization eliminates almost all of the welfare loss obtained under stabilization policies for headline CPI, core CPI, or PPI. In other words, the loss coming from a simple stabilization of RPI compared with the Ramsey optimal solution is negligible.

Regarding the ranking of stabilization policies for other indices, there is no common tendency applicable to all countries. Therefore, one should not ignore the price stickiness and elasticity components of RPI and prefer CPI or PPI.

Steady-state efficiency represents the key assumption that substantially simplifies the analysis. Relaxing this assumption would give the central bank an additional incentive to stabilize one sector rather than another to influence their equilibrium relative price. Extending the analysis in this direction represents a fruitful area of future research.

I abstract from input-output networks across different sectors in the economy. Adding this feature would result in a different formula for the RPI.

Table 5: Welfare loss from simple policy rules

| | Welfare Loss (0.01%) | | | | Ranking | | | |
|---|---|---|---|---|---|---|---|---|
| Country | Ramsey | Headline | Core CPI | PPI | Best | 2nd | 3rd | Worst |
| AUS | .001 | .007 | .016 | .004 | Ramsey | PPI | Headline | Core CPI |
| AUT | .002 | .099 | .099 | .074 | Ramsey | PPI | Core CPI | Headline |
| BEL | .009 | .177 | .261 | .51 | Ramsey | Headline | Core CPI | PPI |
| BGR | .016 | .073 | .168 | .134 | Ramsey | Headline | PPI | Core CPI |
| BRA | 0 | .005 | .004 | .005 | Ramsey | Core CPI | Headline | PPI |
| CAN | .002 | .044 | .06 | .025 | Ramsey | PPI | Headline | Core CPI |
| CHN | 0 | .006 | .006 | .004 | Ramsey | PPI | Headline | Core CPI |
| CYP | .008 | .053 | .048 | .106 | Ramsey | Core CPI | Headline | PPI |
| CZE | .005 | .215 | .253 | .147 | Ramsey | PPI | Headline | Core CPI |
| DEU | .001 | .043 | .038 | .033 | Ramsey | PPI | Core CPI | Headline |
| DNK | .002 | .095 | .046 | .083 | Ramsey | Core CPI | PPI | Headline |
| ESP | .002 | .073 | .03 | .086 | Ramsey | Core CPI | Headline | PPI |
| EST | .006 | .363 | .507 | .324 | Ramsey | PPI | Headline | Core CPI |
| FIN | .002 | .064 | .104 | .091 | Ramsey | Headline | PPI | Core CPI |
| FRA | .001 | .056 | .022 | .058 | Ramsey | Core CPI | Headline | PPI |
| GBR | 0 | .012 | .01 | .012 | Ramsey | Core CPI | Headline | PPI |
| GRC | .002 | .028 | .015 | .035 | Ramsey | Core CPI | Headline | PPI |
| HUN | .003 | .255 | .167 | .159 | Ramsey | PPI | Core CPI | Headline |
| IDN | .001 | .033 | .032 | .036 | Ramsey | Core CPI | Headline | PPI |
| IND | .001 | .012 | .007 | .016 | Ramsey | Core CPI | Headline | PPI |
| IRL | .005 | .332 | .323 | .241 | Ramsey | PPI | Core CPI | Headline |
| ITA | .001 | .071 | .031 | .075 | Ramsey | Core CPI | Headline | PPI |
| JPN | 0 | .008 | .003 | .007 | Ramsey | Core CPI | PPI | Headline |
| KOR | .002 | .181 | .087 | .36 | Ramsey | Core CPI | Headline | PPI |
| LTU | .011 | .053 | .099 | .177 | Ramsey | Headline | Core CPI | PPI |
| LUX | .03 | 1.343 | 1.466 | .932 | Ramsey | PPI | Headline | Core CPI |
| LVA | .007 | .102 | .128 | .101 | Ramsey | PPI | Headline | Core CPI |
| MEX | .001 | .011 | .015 | .008 | Ramsey | PPI | Headline | Core CPI |
| MLT | .008 | 1.671 | .77 | 1.342 | Ramsey | Core CPI | PPI | Headline |
| NLD | .007 | .215 | .085 | .451 | Ramsey | Core CPI | Headline | PPI |
| POL | .001 | .023 | .027 | .022 | Ramsey | PPI | Headline | Core CPI |
| PRT | .001 | .122 | .034 | .148 | Ramsey | Core CPI | Headline | PPI |
| ROU | .003 | .018 | .058 | .015 | Ramsey | PPI | Headline | Core CPI |
| RUS | 0 | .012 | .013 | .008 | Ramsey | PPI | Headline | Core CPI |
| SVK | .01 | .193 | .266 | .27 | Ramsey | Headline | Core CPI | PPI |
| SVN | .008 | .216 | .279 | .15 | Ramsey | PPI | Headline | Core CPI |
| SWE | .001 | .078 | .089 | .159 | Ramsey | Headline | Core CPI | PPI |
| TUR | .001 | .036 | .01 | .035 | Ramsey | Core CPI | PPI | Headline |
| TWN | .003 | .199 | .122 | .162 | Ramsey | Core CPI | PPI | Headline |
| USA | 0 | .015 | .003 | .015 | Ramsey | Core CPI | Headline | PPI |

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# Appendix

## A Proofs and Derivations for Section 2

### A.1 Derivation of equations (13) and (14)

I derive equations (13) and (14) replicated here

$$\frac{P_{s,t}}{P_t} = \frac{P_{s,t-1}}{P_{t-1}} \frac{1}{\Pi_t} \lambda_s \left(1 - (1-\lambda_s)\left(\frac{\tilde{K}_{s,t}}{\tilde{F}_{s,t}}\right)^{1-\theta_s}\right)^{\frac{1}{\theta_s - 1}}$$

$$\frac{P^X_{s,t}}{P_t} = \frac{P^X_{s,t-1}}{P_{t-1}} \frac{1}{\Pi_t} \lambda_s \left(1 - (1-\lambda_s)\left(\frac{\tilde{K}^X_{s,t}}{\tilde{F}^X_{s,t}}\right)^{1-\theta_s}\right)^{\frac{1}{\theta_s - 1}}$$

from the following conditions.

1. Optimal pricing problem of individual firms in equation (12) replicated here

$$\begin{aligned}\left(P_{sit}(0), P^X_{sit}(0)\right) = \arg\max_{(P,P^X)} \sum_{\tau=0}^{\infty} \lambda_s^\tau E_t \Bigg[ \frac{\mathcal{E}_t}{\mathcal{E}_{t+\tau}} \mathcal{M}^*_{t,t+\tau} \\ \times \Bigg\{ \left( (1-\tau_s) P - \left(\frac{\mathcal{E}_{t+\tau} Q^*_{s,t+\tau}}{\alpha_{sm}}\right)^{\alpha_{sm}} \left(\frac{W_{t+\tau}}{\alpha_{sl}}\right)^{\alpha_{sl}} Z^{-1}_{s,t+\tau}\right) \left(\frac{P}{P_{s,t+\tau}}\right)^{-\theta_s} Y_{s,t+\tau} \\ + \left( \left(1-\tau^X_s\right) P^X - \left(\frac{\mathcal{E}_{t+\tau} Q^*_{s,t+\tau}}{\alpha_{sm}}\right)^{\alpha_{sm}} \left(\frac{W_{t+\tau}}{\alpha_{sl}}\right)^{\alpha_{sl}} Z^{-1}_{s,t+\tau}\right) \left(\frac{P^X}{P^X_{s,t+\tau}}\right)^{-\theta_X} Y^X_{s,t+\tau} \Bigg\} \Bigg]\end{aligned}$$

2. The household's condition

$$\mathcal{M}^*_{0t} = \beta^t \frac{\mathcal{E}_t}{P_t} \frac{C_t^{-\sigma}}{\Lambda}$$

derived from the definition of $C^*_t$ and $P^*_t$ given in equation (7) and the risk sharing condition (8)

3. Aggregate price dynamics connecting the sectoral price to the price in the previous period and the newly set price $P_{sit}(0), P^X_{sit}(0)$

$$P_{s,t} = \left(\lambda_s (P_{s,t-1})^{1-\theta_s} + (1-\lambda_s) P_{s,t}(0)^{1-\theta_s}\right)^{\frac{1}{1-\theta_s}}$$

$$P^X_{s,t} = \left(\lambda_s \left(P^X_{s,t-1}\right)^{1-\theta_s} + (1-\lambda_s) P^X_{s,t}(0)^{1-\theta_s}\right)^{\frac{1}{1-\theta_s}},$$

which follows from the aggregation (11) and the i.i.d. likelihood of resetting prices.

The derivation closely follows that in Benigno and Woodford [2005].

First, take the first-order conditions of the pricing problem. The first-order conditions are

$$
\begin{aligned}
&P\sum_{\tau=0}^{\infty}\lambda_s^{\tau}E_t\left[\frac{\mathcal{E}_t}{\mathcal{E}_{t+\tau}}\mathcal{M}^*_{t,t+\tau}P^{\theta_s}_{s,t+\tau}Y_{s,t+\tau}\right]\\
&=(1-\tau_s)^{-1}\,\frac{\theta_s}{\theta_s-1}\sum_{\tau=0}^{\infty}\lambda_s^{\tau}E_t\left[\frac{\mathcal{E}_t}{\mathcal{E}_{t+\tau}}\mathcal{M}^*_{t,t+\tau}\left(\frac{\mathcal{E}_{t+\tau}Q^*_{s,t+\tau}}{\alpha_{sm}}\right)^{\alpha_{sm}}\left(\frac{W_{t+\tau}}{\alpha_{sl}}\right)^{\alpha_{sl}}Z^{-1}_{s,t+\tau}P^{\theta_s}_{s,t+\tau}Y_{s,t+\tau}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
&P^X\sum_{\tau=0}^{\infty}\lambda_s^{\tau}E_t\left[\frac{\mathcal{E}_t}{\mathcal{E}_{t+\tau}}\mathcal{M}^*_{t,t+\tau}\left(P^X_{s,t+\tau}\right)^{\theta_s}Y^X_{s,t+\tau}\right]\\
&=\left(1-\tau_s^X\right)^{-1}\frac{\theta_s}{\theta_s-1}\sum_{\tau=0}^{\infty}\lambda_s^{\tau}E_t\left[\frac{\mathcal{E}_t}{\mathcal{E}_{t+\tau}}\mathcal{M}^*_{t,t+\tau}\left(\frac{\mathcal{E}_{t+\tau}Q^*_{s,t+\tau}}{\alpha_{sm}}\right)^{\alpha_{sm}}\left(\frac{W_{t+\tau}}{\alpha_{sl}}\right)^{\alpha_{sl}}Z^{-1}_{s,t+\tau}\left(P^X_{s,t+\tau}\right)^{\theta_s}Y^X_{s,t+\tau}\right].
\end{aligned}
$$

Using the household's condition $\mathcal{M}^*_{0t}=\beta^t\frac{\mathcal{E}_t}{P_t}\frac{C_t^{-\sigma}}{\Lambda}$,

$$
\begin{aligned}
&P\sum_{\tau=0}^{\infty}\lambda_s^{\tau}E_t\left[\frac{C^{-\sigma}_{t+\tau}}{P_{t+\tau}}P^{\theta_s}_{s,t+\tau}Y_{s,t+\tau}\right]\\
&=(1-\tau_s)^{-1}\,\frac{\theta_s}{\theta_s-1}\sum_{\tau=0}^{\infty}\lambda_s^{\tau}E_t\left[\frac{C^{-\sigma}_{t+\tau}}{P_{t+\tau}}\left(\frac{\mathcal{E}_{t+\tau}Q^*_{s,t+\tau}}{\alpha_{sm}}\right)^{\alpha_{sm}}\left(\frac{W_{t+\tau}}{\alpha_{sl}}\right)^{\alpha_{sl}}Z^{-1}_{s,t+\tau}P^{\theta_s}_{s,t+\tau}Y_{s,t+\tau}\right].
\end{aligned}
$$

$$
\begin{aligned}
&P^X\sum_{\tau=0}^{\infty}\lambda_s^{\tau}E_t\left[\frac{C^{-\sigma}_{t+\tau}}{P_{t+\tau}}\left(P^X_{X,t+\tau}\right)^{\theta_M}Y^X_{X,t+\tau}\right]\\
&=\left(1-\tau_s^X\right)^{-1}\frac{\theta_s}{\theta_s-1}\sum_{\tau=0}^{\infty}\lambda_s^{\tau}E_t\left[\frac{C^{-\sigma}_{t+\tau}}{P_{t+\tau}}\left(\frac{\mathcal{E}_{t+\tau}Q^*_{s,t+\tau}}{\alpha_{sm}}\right)^{\alpha_{sm}}\left(\frac{W_{t+\tau}}{\alpha_{sl}}\right)^{\alpha_{sl}}Z^{-1}_{s,t+\tau}\left(P^X_{s,t+\tau}\right)^{\theta_s}Y^X_{s,t+\tau}\right].
\end{aligned}
$$

Thus, for each sector $s\in S$

$$
\begin{aligned}
\frac{P_{s,t}(0)}{P_{s,t}}=&\left(\underbrace{\sum_{\tau=0}^{\infty}\lambda_s^{\tau}\beta^{\tau}E_t\left[\overbrace{C^{-\sigma}_{t+\tau}\frac{P_{s,t+\tau}}{P_{t+\tau}}\left(\frac{P_{s,t+\tau}}{P_{s,t}}\right)^{\theta_s-1}Y_{s,t+\tau}}^{=:F_{s,t,t+\tau}}\right]}_{=:\tilde{F}_{s,t}}\right)^{-1}\\
&\times\underbrace{\sum_{\tau=0}^{\infty}\lambda_s^{\tau}\beta^{\tau}E_t\left[\overbrace{(1-\tau_s)^{-1}\frac{\theta_s}{\theta_s-1}C^{-\sigma}_{t+\tau}\left(\frac{Q_{t+\tau}Q^*_{s,t+\tau}}{\alpha_{sm}P^*_{t+\tau}}\right)^{\alpha_{sm}}\left(\frac{W_{t+\tau}}{\alpha_{sl}P_{t+\tau}}\right)^{\alpha_{sl}}Z^{-1}_{s,t+\tau}\left(\frac{P_{s,t+\tau}}{P_{s,t}}\right)^{\theta_s}Y_{s,t+\tau}}^{=:K_{s,t,t+\tau}}\right]}_{=:\tilde{K}_{s,t}}
\end{aligned}
$$

and for exports

$$\frac{P_{s,t}^{X}(0)}{P_{s,t}} = \left( \underbrace{\sum_{\tau=0}^{\infty} \lambda_s^{\tau} \beta^{\tau} E_t \left[ \overbrace{C_{t+\tau}^{-\sigma} \frac{P_{s,t+\tau}^{X}}{P_{t+\tau}} \left( \frac{P_{s,t+\tau}^{X}}{P_{s,t}^{X}} \right)^{\theta_s - 1} Y_{s,t+\tau}^{X}}^{=:F_{s,t,t+\tau}^{X}} \right]}_{=:\tilde{F}_{s,t}^{X}} \right)^{-1} .$$

$$\times \underbrace{\sum_{\tau=0}^{\infty} \lambda_s^{\tau} \beta^{\tau} E_t \left[ \overbrace{\left(1-\tau_s^X\right)^{-1} \frac{\theta_s}{\theta_s - 1} C_{t+\tau}^{-\sigma} \left( \frac{Q_{t+\tau} Q_{s,t+\tau}^{*}}{\alpha_{sm} P_{t+\tau}^{*}} \right)^{\alpha_{sm}} \left( \frac{W_{t+\tau}}{\alpha_{sl} P_{t+\tau}} \right)^{\alpha_{sl}} Z_{s,t+\tau}^{-1} \left( \frac{P_{s,t+\tau}^{X}}{P_{s,t}^{X}} \right)^{\theta_s} Y_{s,t+\tau}^{X}}^{=:K_{s,t,t+\tau}^{X}} \right]}_{=:\tilde{K}_{s,t}^{X}} .$$

Next, rewrite the dynamics and insert the above conditions:

$$\begin{aligned} 1 &= \left( \lambda_s \left( \frac{P_{s,t-1}}{P_{s,t}} \right)^{1-\theta_s} + (1-\lambda_s) \left( \frac{P_{s,t}(0)}{P_t} \frac{P_t}{P_{s,t}} \right)^{1-\theta_s} \right)^{\frac{1}{1-\theta_s}} \\ &= \left( \lambda_s \left( \frac{P_{s,t-1}}{P_{s,t}} \right)^{1-\theta_s} + (1-\lambda_s) \left( \frac{\tilde{K}_{s,t}}{\tilde{F}_{s,t}} \right)^{1-\theta_s} \right)^{\frac{1}{1-\theta_s}} . \end{aligned}$$

$$\begin{aligned} 1 &= \left( \lambda_s \left( \frac{P_{s,t-1}^{X}}{P_{s,t}^{X}} \right)^{1-\theta_s} + (1-\lambda_s) \left( \frac{P_{s,t}^{X}(0)}{P_t} \frac{P_t}{P_{s,t}^{X}} \right)^{1-\theta_s} \right)^{\frac{1}{1-\theta_s}} \\ &= \left( \lambda_s \left( \frac{P_{s,t-1}^{X}}{P_{s,t}^{X}} \right)^{1-\theta_s} + (1-\lambda_s) \left( \frac{\tilde{K}_{s,t}^{X}}{\tilde{F}_{s,t}^{X}} \right)^{1-\theta_s} \right)^{\frac{1}{1-\theta_s}} . \end{aligned}$$

By rearranging this, we can obtain equation (13) and (14).

Finally, note that under the assumption of a bounded solution,

$$\tilde{F}_{s,t} = \sum_{\tau=0}^{\infty} \lambda_s^{\tau} \beta^{\tau} E_t F_{s,t,t+\tau}$$

can equivalently be written as

$$\begin{aligned} \tilde{F}_{s,t} &= F_{s,t,t+1} + \lambda_s \beta \sum_{\tau=0}^{\infty} \lambda_s^{\tau} \beta^{\tau} E_t F_{s,t,t+1+\tau} \\ &= F_{s,t,t} + \lambda_s \beta \left(\Pi_{s,t+1}\right)^{\theta_s - 1} E_t \tilde{F}_{s,t+1}. \end{aligned}$$

Similarly, for $\tilde{K}_{s,t}$, $\tilde{F}_{s,t}^{X}$, $\tilde{K}_{s,t}^{X}$. Thus we obtain the equivalent definitions given in equation (15)-(18).

## A.2 Derivation of equation (20)

I derive the aggregate resource constraint (20) replicated here

$$Z_{st}L_{st}=\left(\frac{\alpha_{sl}}{\alpha_{sm}}\frac{Q_tQ^*_{st}/P^*_t}{W_t/P_t}\right)^{\alpha_{sm}}\left(\Delta_{st}C_{st}+\Delta^X_{st}X_{st}\right),\ M_{st}=\frac{\alpha_{sm}}{\alpha_{sl}}\frac{W_t/P_t}{Q_tQ^*_{st}/P^*_t}L_{st}$$

together with the evolution of the price dispersion (21) and (22) from the following conditions.

1. Market clearing conditions

$$\sum_{s\in S}\int L_{sit}di=L_t,\int M_{sit}di=M_{st}$$

2. Factor demand from firm's optimization conditions

$$M_{it}=\left(\frac{\alpha_{sm}}{\alpha_{sl}}\frac{W_t}{\mathcal{E}_tQ^*_{st}}\right)^{\alpha_{sl}}\left(\left(\frac{P_{sit}}{P_{st}}\right)^{-\theta_s}\frac{Y_{st}}{Z_{st}}+\left(\frac{P^X_{sit}}{P^X_{st}}\right)^{-\theta_s}\frac{Y^X_{st}}{Z_{st}}\right)$$

$$L_{it}=\left(\frac{\alpha_{sl}}{\alpha_{sm}}\frac{\mathcal{E}_tQ^*_{st}}{W_t}\right)^{\alpha_{sm}}\left(\left(\frac{P_{sit}}{P_{st}}\right)^{-\theta_s}\frac{Y_{st}}{Z_{st}}+\left(\frac{P^X_{sit}}{P^X_{st}}\right)^{-\theta_s}\frac{Y^X_{st}}{Z_{st}}\right)$$

3. Optimal pricing equation obtained in Appendix A.1.

$$\frac{P_{s,t}(0)}{P_t}\frac{P_t}{P_{s,t}}=\frac{\tilde{K}_{s,t}}{\tilde{F}_{s,t}}=\left(\frac{1-\lambda_s\left(\frac{P_{s,t-1}}{P_{s,t}}\right)^{1-\theta_s}}{1-\lambda_s}\right)^{\frac{1}{1-\theta_s}}.$$

The derivation here closely follows that in Benigno and Woodford [2005].

To obtain the aggregate resource constraint (20) , combine these conditions 1 and 2. Then,

$$L_{st}=\int L_{sit}di=\left(\frac{\alpha_{sl}}{\alpha_{sm}}\frac{\mathcal{E}_tQ^*_{st}}{W_t}\right)^{\alpha_{sm}}\left(\underbrace{\int\left(\frac{P_{sit}}{P_{st}}\right)^{-\theta_s}di}_{:=\Delta_{st}}\frac{Y_{st}}{Z_{st}}+\underbrace{\int\left(\frac{P^X_{sit}}{P^X_{st}}\right)^{-\theta_s}di}_{:=\Delta^X_{st}}\frac{Y^X_{st}}{Z_{st}}\right),$$

$$M_{st}=\frac{\alpha_{sm}}{\alpha_{sl}}\frac{W_t/P_t}{Q_tQ^*_{st}/P^*_t}L_{st}.$$

Note that the second condition also uses the definition of the real exchange rate $Q_t\equiv\frac{\mathcal{E}_tP_t*}{P_t}$.

To obtain the dynamics of price dispersion, rewrite the definition of the dispersion using the optimal pricing equation as follows.

$$
\begin{aligned}
\Delta_{st} &= \int \left(\frac{P_{sit}}{P_{st}}\right)^{-\theta_s} di \\
&= \lambda_s \int \left(\frac{P_{sit-1}}{P_{st}}\right)^{-\theta_s} di + (1-\lambda_s) \int \left(\frac{P_{sit}(0)}{P_{st}}\right)^{-\theta_s} di \\
&= \lambda_s \left(\frac{P_{st-1}}{P_{st}}\right)^{-\theta_s} \int \left(\frac{P_{sit-1}}{P_{st-1}}\right)^{-\theta_s} di + (1-\lambda_s) \left(\frac{1-\lambda_s\left(\frac{P_{s,t-1}}{P_{s,t}}\right)^{1-\theta_s}}{1-\lambda_s}\right)^{\frac{-\theta_s}{1-\theta_s}} \\
&= \lambda_s \left(\frac{P_{st-1}}{P_{st}}\right)^{-\theta_s} \Delta_{s,t-1} + (1-\lambda_s) \left(\frac{1-\lambda_s\left(\frac{P_{s,t-1}}{P_{s,t}}\right)^{1-\theta_s}}{1-\lambda_s}\right)^{\frac{-\theta_s}{1-\theta_s}} \\
&= \lambda_s \left(\frac{P_{st}}{P_{st-1}}\right)^{\theta_s} \Delta_{s,t-1} + (1-\lambda_s) \left(\frac{1-\lambda_s\left(\frac{P_{s,t}}{P_{s,t-1}}\right)^{\theta_s-1}}{1-\lambda_s}\right)^{\frac{\theta_s}{\theta_s-1}} .
\end{aligned}
$$

# B Proofs and Derivations for Section 3

## B.1 Planner's solution

Given $\left\{\frac{Q^*_{Mt}}{P^*_t}, \frac{P^*_{Xt}}{P^*_t}, \mathcal{M}^*_{0,t}\right\}_{t=0}^{\infty}, \Lambda$, the planner maximizes

$$
\max_{D_0, \left\{(C_{st}, M_{st}, X_{st}, L_{st})_{s\in S}\right\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{\left(\prod_{s\in S} C_{st}^{\psi_s}\right)^{1-\sigma}}{1-\sigma} - \frac{\left(\sum_{s\in S} L_{st}\right)^{1+\phi}}{1+\phi}\right] + \Lambda D_0,
$$

$$
s.t. \begin{cases} Z_{s,t} M_{st}^{\alpha_{sm}} L_{st}^{\alpha_{sl}} = C_{st} + X_{st} \ \forall s \in S & [\lambda_{st}] \\ E_0 \sum_{t=0}^{\infty} \left[\mathcal{M}^*_{0,t} P^*_t \sum_{s\in S} \left(X_{st}^{\frac{\theta^*_s - 1}{\theta^*_s}} X_{st}^{*\frac{1}{\theta^*_s}} \frac{P^*_{st}}{P^*_t} - \frac{Q^*_{st}}{P^*_t} M_{st}\right)\right] = D_0 & [\lambda_D] \end{cases} .
$$

The first-order conditions are

$$
\begin{cases}
[C_{st}] & \beta^t C_t^{1-\sigma} \frac{\psi_s}{C_{st}} = \lambda_{st} \\
[M_{st}] & \alpha_{sm} \frac{Y_{st}}{M_{st}} \lambda_{st} = \mathcal{M}^*_{0,t} Q^*_{st} \lambda_D \\
[X_{st}] & \lambda_{st} = \mathcal{M}^*_{0,t} \frac{P^X_{st}}{\mathcal{E}_t} \frac{\theta^*_s - 1}{\theta^*_s} \lambda_D \\
[L_{st}] & \beta^t L_t^{\phi} = \alpha_{sl} \frac{Y_{st}}{L_{st}} \lambda_{st} \\
[D_0] & \Lambda = \lambda_D
\end{cases}
$$

From the first-order conditions, we obtain aggregate consumption and the consumption price index. Rearranging the FOC with respect to $C_{st}$,

$$
C_{st} = \beta^t C_t^{1-\sigma} \frac{\psi_s}{\lambda_{st}}.
$$

Plugging this into the consumption aggregator $C_t = \prod_{s\in S} C_{st}^{\psi_s}$, we obtain

$$C_t = \beta^t C_t^{1-\sigma} \prod_{s\in S} \psi_s^{\psi_s} \prod_{s\in S} \left(\frac{1}{\lambda_{st}}\right)^{\psi_s}.$$

Multiplying both sides by $\mathcal{M}_{0,t}^* \lambda_D = \mathcal{M}_{0,t}^* \Lambda$, we have

$$\mathcal{M}_{0,t}^* \Lambda = \beta^t C_t^{-\sigma} \prod_{s\in S} \psi_s^{\psi_s} \prod_{s\in S} \left(\frac{\mathcal{M}_{0,t}^* \Lambda}{\lambda_{st}}\right)^{\psi_s} = \beta^t C_t^{-\sigma} \left(\frac{P_t}{\mathcal{E}_t}\right)^{-1},$$

where $\frac{P_t}{\mathcal{E}_t} := \prod_{s\in S} \psi_s^{-\psi_s} \prod_{s\in S} \left(\frac{\lambda_{st}}{\mathcal{M}_{0,t}^* \Lambda}\right)^{\psi_s} = \prod_{s\in S} \psi_s^{-\psi_s} \prod_{s\in S} \left(\frac{\beta^t C_t^{1-\sigma} \frac{\psi_s}{C_{st}}}{\mathcal{M}_{0,t}^* \Lambda}\right)^{\psi_s}$ is defined as the shadow price of the aggregate consumption in terms of international currency. Combining this with the assumption on the relationship between $\mathcal{M}_{0,t}^*$ , $C_t^*$ and $P_t^*$, we obtain the risk sharing condition

$$\beta^t \frac{\left(C_t^*\right)^{-\sigma} / P_t^*}{\left(C_0^*\right)^{-\sigma} / P_0^*} \Lambda = \beta^t C_t^{-\sigma} \mathcal{E}_t P_t^{-1} \Rightarrow C_t = \xi C_t^* Q_t^{\frac{1}{\sigma}},$$

where $\xi$ is the same constant as that in equation (8). The real exchange rate here is defined as $Q_t = \frac{\mathcal{E}_t P_t^*}{P_t}$ using the shadow price of the aggregate consumption defined above.

We can also obtain the intra-temporal conditions. Due to the assumption $\alpha_{sl} > 0$ for all $s \in S$, combining the FOC with respect to $C_{st}$ and that with respect to $L_{st}$ leads to

$$C_t \frac{\psi_s}{C_{st}} \alpha_{sl} \frac{Y_{st}}{L_{st}} = \frac{L_t^{\phi}}{C_t^{-\sigma}}.$$

For those sectors with $\alpha_{sm} > 0$, combining the FOC with respect to $M_{st}$ and that with respect to $L_{st}$,

$$\frac{\alpha_{sl} \frac{Y_{st}}{L_{st}}}{\alpha_{sm} \frac{Y_{st}}{M_{st}}} = \frac{\beta^t L_t^{\phi}}{\mathcal{M}_{0,t}^* Q_{st}^* \Lambda} = \frac{L_t^{\phi} / C_t^{-\sigma}}{Q_t \frac{Q_{st}^*}{P_t^*}}.$$

From this, we can calculate the aggregate labor productivity:

$$\begin{aligned} Y_{st} &= Z_{s,t} M_{st}^{\alpha_{sm}} L_{st}^{\alpha_{sl}} \\ &= Z_{s,t} \left(\alpha_{sm} \frac{L_t^{\phi} / C_t^{-\sigma}}{Q_t \frac{Q_{st}^*}{P_t^*}} \frac{1}{\alpha_{sl} \frac{1}{L_{st}}}\right)^{\alpha_{sm}} L_{st}^{\alpha_{sl}} \\ &= Z_{s,t} \left(\frac{\alpha_{sm}}{\alpha_{sl}} \frac{L_t^{\phi}}{C_t^{-\sigma}} \frac{P_t^*}{Q_t Q_{st}^*}\right)^{\alpha_{sm}} L_{st} \end{aligned}$$

For those sectors with positive exports, combining the FOC with respect to $X_{st}$ and that with respect to $C_{st}$, we have

$$\frac{\theta_s^* - 1}{\theta_s^*} Q_t \frac{P_{st}^*}{P_t^*} \left(\frac{X_{st}}{X_{st}^*}\right)^{-\frac{1}{\theta_s^*}} = \frac{P_{st}^X}{P_t} \frac{\theta_s^* - 1}{\theta_s^*} = C_t \frac{\psi_s}{C_{st}}$$

Combining these, we obtain the conditions in equations (24)-(27).

## B.2 Flexible price equilibrium

The household.

The first-order conditions are

$$\begin{aligned}\beta^t \psi_s \tfrac{C_t^{1-\sigma}}{C_{st}} &= \tfrac{\mathcal{M}_{0t}^*}{\mathcal{E}_t}\lambda P_{st}\\ \beta^t L_t^{\phi} &= \tfrac{\mathcal{M}_{0t}^*}{\mathcal{E}_t}\lambda W_{st}\\ \Lambda &= \lambda\end{aligned}$$

From the linearity of labor aggregator, we can immediately see that$W_{st} = W_s$ must hold in the equilibrium. From the first-order conditions, we can calculate aggregate consumption and price index.

$$C_{st} = \beta^t \psi_s \frac{C_t^{1-\sigma}}{\frac{\mathcal{M}_{0t}^*}{\mathcal{E}_t}\lambda P_{st}} \Rightarrow C_t = \prod_{s\in S} C_{st}^{\psi_s} = \beta^t \frac{C_t^{1-\sigma}}{\frac{\mathcal{M}_{0t}^*}{\mathcal{E}_t}\lambda} \prod_{s\in S}\left(\frac{\psi_s}{P_{st}}\right)^{\psi_s}$$

$$\Rightarrow \mathcal{M}_{0t}^* = \beta^t \mathcal{E}_t \frac{C_t^{-\sigma}}{\lambda}\prod_{s\in S}\left(\frac{\psi_s}{P_{st}}\right)^{\psi_s} = \beta^t \frac{\mathcal{E}_t}{P_t}\frac{C_t^{-\sigma}}{\Lambda},$$

where $P_t = \prod_{s\in S}\left(\frac{P_{st}}{\psi_s}\right)^{\psi_s}$ is the consumer price index. Combining this with the same sequence of $C_t^*$ and $P_t^*$as in the planner's problem,

$$\beta^t \frac{\left(C_t^*\right)^{-\sigma}/P_t^*}{\left(C_0^*\right)^{-\sigma}/P_0^*}\Lambda = \beta^t C_t^{-\sigma}\mathcal{E}_t P_t^{-1} \Rightarrow C_t = \xi C_t^* Q_t^{\frac{1}{\sigma}},$$

where $\xi = \left(\frac{\Lambda}{\left(C_0^*\right)^{-\sigma}/P_0^*}\right)^{-\frac{1}{\sigma}}$ is the same constant as the planner's problem as long as the marginal utility $\Lambda$of the initial debt is the same.

We also get intra-temporal conditions

$$\begin{aligned}\psi_s C_t &= \tfrac{P_{st}}{P_t} C_{st}\\ \tfrac{L_t^{\phi}}{C_t^{-\sigma}} &= \tfrac{W_t}{P_t}\end{aligned}.$$

The aggregator firm.

There are two aggregator firms in each sector: one for domestically consumed goods and the other for exported goods. The variables related to exports are indicated by the superscript $X$. The sectoral aggregator firm's cost minimization for domestic use is for each $s \in S$,

$$\min_{\{Y_{sit}\}_i} \int P_{sit} Y_{sit} di \ s.t.\ Y_{st} = \left(\int Y_{sit}^{\frac{\theta_s - 1}{\theta_s}} di\right)^{\frac{\theta_s}{\theta_s - 1}}$$

$$\Rightarrow Y_{sit} = \left(\frac{P_{sit}}{P_{st}}\right)^{-\theta_s} Y_{st},\ P_{st} = \left(\int P_{sit}^{1-\theta_s} di\right)^{\frac{1}{1-\theta_s}}$$

and for export goods,

$$\min_{\left\{Y_{sit}^X\right\}_i} \int P_{sit}^X Y_{sit}^X di \ s.t.\ Y_{st}^X = \left(\int \left(Y_{sit}^X\right)^{\frac{\theta_s - 1}{\theta_s}} di\right)^{\frac{\theta_s}{\theta_s - 1}}$$

$$\Rightarrow Y_{sit}^X = \left(\frac{P_{sit}^X}{P_{st}^X}\right)^{-\theta_s} Y_{st}^X,\ P_{st}^X = \left(\int \left(P_{sit}^X\right)^{1-\theta_s} di\right)^{\frac{1}{1-\theta_s}}.$$

The Individual Firm.

The individual firm in sector $s \in S$ takes wage $W_t$, import price $\mathcal{E}_t Q_{st}^*$, the demand function derived above, production function and tax rates for domestic sales$\tau_s$ and foreign sales $\tau_s^X$ as given. I allow the firm to set different prices for domestic consumers $P_{sit}$ and for foreign buyers $P_{sit}^X$ (pricing to market). As we will see later, this is necessary for the flexible price equilibrium to be efficient.

$$\max_{P_{sit}, P_{sit}^X, L_{sit}, M_{sit}, Y_{sit}, Y_{sit}^X} (1-\tau_s) P_{sit} Y_{sit} + \left(1-\tau_s^X\right) P_{sit}^X Y_{sit}^X - W_t L_{sit} - \mathcal{E}_t Q_{st}^* M_{sit}$$

$$s.t. \begin{cases} Y_{sit} = \left(\frac{P_{sit}}{P_{st}}\right)^{-\theta_s} Y_{st} \\ Y_{sit}^X = \left(\frac{P_{sit}^X}{P_{st}^X}\right)^{-\theta_s} Y_{st}^X \\ Y_{sit} + Y_{sit}^X = Z_{s,t} M_{sit}^{\alpha_{sm}} L_{sit}^{\alpha_{sl}} \end{cases}.$$

Solving the cost minimization problem as its sub-problem, the marginal cost can be calculated as $\left(\frac{\mathcal{E}_t Q_{st}^*}{\alpha_{sm}}\right)^{\alpha_{sm}} \left(\frac{W_t}{\alpha_{sl}}\right)^{\alpha_{sl}} Z_{st}^{-1}$ and the factor demand should satisfy

$$M_{sit} = \frac{\alpha_{sm}}{\alpha_{sl}} \frac{W_t}{\mathcal{E}_t Q_{st}^*} L_{sit}.$$

Thus,

$$\Rightarrow \max_{P_{sit}, P_{sit}^X} (1-\tau_s) P_{sit} \left(\frac{P_{sit}}{P_{st}}\right)^{-\theta_s} Y_{st} + \left(1-\tau_s^X\right) P_{sit}^X \left(\frac{P_{sit}^X}{P_{st}^X}\right)^{-\theta_s} Y_{st}^X$$
$$- \left(\frac{\mathcal{E}_t Q_{st}^*}{\alpha_{sm}}\right)^{\alpha_{sm}} \left(\frac{W_t}{\alpha_{sl}}\right)^{\alpha_{sl}} Z_{st}^{-1} \left\{ \left(\frac{P_{sit}}{P_{st}}\right)^{-\theta_s} Y_{st} + \left(\frac{P_{sit}^X}{P_{st}^X}\right)^{-\theta_s} Y_{st}^X \right\}$$

The first-order conditions are

$$\begin{cases} (1-\theta_s)(1-\tau_s) P_{sit} \frac{Y_{sit}}{P_{sit}} + \theta_s \left(\frac{\mathcal{E}_t Q_{st}^*}{\alpha_{sm}}\right)^{\alpha_{sm}} \left(\frac{W_t}{\alpha_{sl}}\right)^{\alpha_{sl}} Z_{st}^{-1} \frac{Y_{sit}}{P_{sit}} = 0 \\ (1-\theta_s)\left(1-\tau_s^X\right) P_{sit}^X \frac{Y_{sit}^X}{P_{sit}^X} + \theta_s \left(\frac{\mathcal{E}_t Q_{st}^*}{\alpha_{sm}}\right)^{\alpha_{sm}} \left(\frac{W_t}{\alpha_{sl}}\right)^{\alpha_{sl}} Z_{st}^{-1} \frac{Y_{sit}^X}{P_{sit}^X} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} P_{sit} = (1-\tau_s)^{-1} \frac{\theta_s}{\theta_s - 1} \left(\frac{\mathcal{E}_t Q_{st}^*}{\alpha_{sm}}\right)^{\alpha_{sm}} \left(\frac{W_t}{\alpha_{sl}}\right)^{\alpha_{sl}} Z_{st}^{-1} \\ P_{sit}^X = \left(1-\tau_s^X\right)^{-1} \frac{\theta_s}{\theta_s - 1} \left(\frac{\mathcal{E}_t Q_{st}^*}{\alpha_{sm}}\right)^{\alpha_{sm}} \left(\frac{W_t}{\alpha_{sl}}\right)^{\alpha_{sl}} Z_{st}^{-1} \end{cases}$$

With flexible prices, all firms are symmetric within a sector. Thus, subscript $i$ can be dropped. In summary, we have

$$\begin{cases} P_{st} = (1-\tau_s)^{-1} \frac{\theta_s}{\theta_s - 1} \left(\frac{\mathcal{E}_t Q_{st}^*}{\alpha_{sm}}\right)^{\alpha_{sm}} \left(\frac{W_t}{\alpha_{sl}}\right)^{\alpha_{sl}} Z_{st}^{-1} \\ P_{st}^X = \left(1-\tau_s^X\right)^{-1} \frac{\theta_s}{\theta_s - 1} \left(\frac{\mathcal{E}_t Q_{st}^*}{\alpha_{sm}}\right)^{\alpha_{sm}} \left(\frac{W_t}{\alpha_{sl}}\right)^{\alpha_{sl}} Z_{st}^{-1} \end{cases}$$

$$\Rightarrow \begin{cases} P_{st} = \underbrace{(1-\tau_s)^{-1} \frac{\theta_s}{\theta_s - 1}}_{=:\chi_s^{-1}} \left(\frac{\mathcal{E}_t Q_{st}^*}{\alpha_{sm}}\right)^{\alpha_{sm}} \left(\frac{W_t}{\alpha_{sl}}\right)^{\alpha_{sl}} Z_{st}^{-1} \\ P_{st}^X = \underbrace{\left(1-\tau_s^X\right)^{-1} (1-\tau_s) \frac{\theta_s^* - 1}{\theta_s^*}}_{=:\nu_s^{-1}} \frac{\theta_s^*}{\theta_s^*-1} P_{st} \end{cases}$$

Combining firms' pricing equations and factor demand equations with the household's optimization condition,

$$\begin{cases} \psi_s C_t = \frac{P_{st}}{P_t} C_{st} \\ \frac{L_t^\phi}{C_t^{-\sigma}} = \frac{W_t}{P_t} \\ P_{st} = \chi_s^{-1} \left(\frac{\mathcal{E}_t Q_{st}^*}{\alpha_{sm}}\right)^{\alpha_{sm}} \left(\frac{W_t}{\alpha_{sl}}\right)^{\alpha_{sl}} Z_{st}^{-1} \\ P_{st}^X = \nu_s^{-1} \frac{\theta_s^*}{\theta_s^*-1} P_{st} \\ M_{st} = \frac{\alpha_{sm}}{\alpha_{sl}} \frac{W_t}{\mathcal{E}_t Q_{st}^*} L_{st} \end{cases} \Leftrightarrow \begin{cases} C_t \frac{\psi_s}{C_{st}} \alpha_{sl} \frac{Z_{st} M_{st}^{\alpha_{sm}} L_{st}^{\alpha_{sl}}}{L_{st}} = \chi_s^{-1} \frac{L_t^\phi}{C_t^{-\sigma}} \\ \frac{\alpha_{sl}}{L_{st}} \frac{M_{st}}{\alpha_{sm}} = \frac{L_t^\phi / C_t^{-\sigma}}{Q_t \frac{Q_{st}^*}{P_t^*}} \\ \frac{\theta_s^*-1}{\theta_s^*} Q_t \frac{P_{st}^*}{P_t^*} \frac{P_{st}^X}{\mathcal{E}_t P_{st}^*} = \nu_s^{-1} C_t \frac{\psi_s}{C_{st}} \end{cases}$$

Recall the assumption on the foreign demand for exports

$$X_{st} = \left(\frac{P_{st}^X}{\mathcal{E}_t P_{st}^*}\right)^{-\theta_s^*} X_{st}^*.$$

Then, the third condition can be equivalently written as

$$\frac{\theta_s^* - 1}{\theta_s^*} Q_t \frac{P_{st}^*}{P_t^*} \left(X_{st}^*\right)^{\frac{1}{\theta_s^*}} = \nu_s^{-1} X_{st}^{\frac{1}{\theta_s^*}} C_t \frac{\psi_s}{C_{st}}.$$

Finally, using production technology and the market clearing condition, $X_{st} = Z_{st} M_{st}^{\alpha_{sm}} L_{st}^{\alpha_{sl}} - C_{st}$. Thus,

$$\frac{\theta_s^* - 1}{\theta_s^*} Q_t \frac{P_{st}^*}{P_t^*} \left(X_{st}^*\right)^{\frac{1}{\theta_s^*}} = \nu_s^{-1} \left(Z_{st} M_{st}^{\alpha_{sm}} L_{st}^{\alpha_{sl}} - C_{st}\right)^{\frac{1}{\theta_s^*}} C_t \frac{\psi_s}{C_{st}}.$$

Combining these leads to equations (28)-(31).

## B.3 Definition of optimal steady state

The optimal steady state is defined as follows.

**Definition 6.** The optimal steady state is the solution to the following problem. Given constant $\left(\left\{\frac{Q_{st}^*}{P_t^*}, \frac{P_{st}^*}{P_t^*}, Z_{st}, X_{st}^*\right\}_{s\in S}, \mathcal{M}_{t+1}^*, P_t^*\right) = \left(\left\{Q_s^*, P_s^*, Z_s, X_s^*\right\}_{s\in S}, \beta, 1\right)$, tax $\left(\tau_s, \tau_s^X\right)_{s\in S}$, and initial state variables $\left(P_{-1}, \mathcal{E}_{-1}, \left\{\Delta_{s,-1}, \Delta_{s,-1}^X\right\}_{s\in S}\right) = \left(1, 1, \{1,1\}_{s\in S}\right)$, the central bank maximizes

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\phi}}{1+\phi}\right] + \Lambda D_0$$

s.t.

$$\begin{cases}\psi_s C_t = \frac{P_{st}}{P_t} C_{st} \\ \frac{L_t^{\phi}}{C_t^{-\sigma}} = \frac{W_t}{P_t} \\ C_t = \xi C_t^* Q_t^{\frac{1}{\sigma}} \\ f_s\left(\frac{P^u_{s,t}}{P_t}, \Pi_t; \frac{P^u_{s,t-1}}{P_{t-1}}\right) \tilde{K}^u_{s,t} = \tilde{F}^u_{s,t} \\ \tilde{K}^u_{s,t} = (1-\tau^u_s)^{-1} \frac{\theta_s}{\theta_s - 1} C_t^{-\sigma} \left(\frac{Q_t Q^*_{s,t}}{\alpha_{sm} P_t^*}\right)^{\alpha_{sm}} \left(\frac{W_t}{\alpha_{sl} P_t}\right)^{\alpha_{sl}} \frac{1}{Z_{st}} Y^u_{s,t} + \lambda_s \beta E_t \left(\Pi^u_{s,t+1}\right)^{\theta_s} \tilde{K}^u_{s,t+1} \\ \tilde{F}^u_{s,t} = C_t^{-\sigma} \frac{P^u_{s,t}}{P_t} Y^u_{s,t} + \lambda_s \beta E_t \left(\Pi^u_{s,t+1}\right)^{\theta_s - 1} \tilde{F}^u_{s,t+1} \\ \Delta^u_{st} = \lambda_s \left(\frac{P^u_{st}}{P^u_{st-1}}\right)^{\theta_s} \Delta^u_{s,t-1} + (1-\lambda_s)\left(f_s\left(\frac{P^u_{s,t}}{P_t}, \Pi_t; \frac{P^u_{s,t-1}}{P_{t-1}}\right)\right)^{\theta_s} \\ Z_{st} L_{st} = \left(\frac{\alpha_{sl}}{\alpha_{sm}} \frac{Q_t Q^*_{st}}{W_t/P_t}\right)^{\alpha_{sm}} \left\{\Delta_{st} C_{st} + \Delta^X_{st} \left(\frac{P^X_{st}/P_t}{Q_t P^*_s}\right)^{-\theta^*_s} X^*_s\right\} \\ M_{st} = \frac{\alpha_{sm}}{\alpha_{sl}} \frac{W_t/P_t}{Q_t Q^*_s} L_{st} \\ E_0 \sum_{t=0}^{\infty} \left[\beta^t \sum_{s\in S} \left(\left(\frac{P^X_{st}/P_t}{Q_t P^*_s}\right)^{-\theta^*_s} X^*_s \frac{P^X_{st}}{Q_t P_t} - M_{st} Q^*_s\right)\right] = D_0 \end{cases}$$

where

$$Y_{s,t} = C_{st},\ Y^X_{s,t} = \left(\frac{P^X_{st}/P_t}{Q_t P^*_s}\right)^{-\theta^*_s} X^*_s$$

and $C_t = \prod_{s\in S} C_{st}^{\psi_s},\ L_t = \sum_{s\in S} L_{st}$.

## B.4 The solution and properties of the optimal steady state

### B.4.1 The solution

Before solving this, solve out $C_{st},\ M_{st}$ as functions of prices and aggregate consumption. Define the Lagrangian

$$
\begin{aligned}
\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \Bigg[ & \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{\left(\sum_{s \in S} L_{st}\right)^{1+\phi}}{1+\phi} \\
& + \sum_{s \in S} \Xi_{1t}^s \left\{ \left( \Delta_{st} \frac{\psi_s C_t}{P_{st}/P_t} + \Delta_{st}^X \left( \frac{P_{st}^X/P_t}{Q_t P_s^*} \right)^{-\theta_s^*} X_s^* \right) \bar{Z}_s^{-1} \left( \frac{\alpha_{sl}}{\alpha_{sm}} \frac{Q_t Q_s^*}{W_t/P_t} \right)^{\alpha_{sm}} - L_{st} \right\} \\
& + \Xi_{2t} \left\{ \left( \sum_{s \in S} L_{st} \right)^{\phi} - \frac{W_t}{P_t} C_t^{-\sigma} \right\} + \Xi_{3t} \left( \frac{P_{st}/P_t}{\psi_s} \right)^{\psi_s} \\
& + \sum_{(s,u)} \Xi_{4t}^{(s,u)} \left\{ f_s \left( \frac{P_{s,t}^u}{P_t}, \Pi_t; \frac{P_{s,t-1}^u}{P_{t-1}} \right) \tilde{K}_{s,t}^u - \tilde{F}_{s,t}^u \right\} \\
& + \sum_{s \in S} \Xi_{5t}^s \left\{ \psi_s C_t^{1-\sigma} + \lambda_s \beta E_t \left( \Pi_{s,t+1} \right)^{\theta_s - 1} \tilde{F}_{s,t+1} - \tilde{F}_{s,t} \right\} \\
& + \sum_{s \in S} \Xi_{5t}^{s,X} \left\{ C_t^{-\sigma} \frac{P_{s,t}^X}{P_t} \left( \frac{P_{st}^X/P_t}{Q_t P_s^*} \right)^{-\theta_s^*} X_s^* + \lambda_s \beta E_t \left( \Pi_{s,t+1}^X \right)^{\theta_s - 1} \tilde{F}_{s,t+1}^X - \tilde{F}_{s,t}^X \right\} \\
& + \sum_{s \in S} \Xi_{6t}^s \Bigg\{ (1-\tau_s)^{-1} \frac{\theta_s}{\theta_s - 1} C_t^{1-\sigma} \left( \frac{Q_t Q_s^*}{\alpha_{sm}} \right)^{\alpha_{sm}} \left( \frac{W_t}{\alpha_{sl} P_t} \right)^{\alpha_{sl}} \frac{1}{Z_{st}} \frac{\psi_s}{P_{st}/P_t} \\
& + \lambda_s \beta E_t \left( \Pi_{s,t+1} \right)^{\theta_s} \tilde{K}_{s,t+1} - \tilde{K}_{s,t} \Bigg\} \\
& + \sum_{s \in S} \Xi_{6t}^{s,X} \Bigg\{ \left( 1 - \tau_s^X \right)^{-1} \frac{\theta_s}{\theta_s - 1} C_t^{-\sigma} \left( \frac{Q_t Q_s^*}{\alpha_{sm}} \right)^{\alpha_{sm}} \left( \frac{W_t}{\alpha_{sl} P_t} \right)^{\alpha_{sl}} \frac{1}{Z_{st}} \left( \frac{P_{st}^X/P_t}{Q_t P_s^*} \right)^{-\theta_s^*} X_s^*. \\
& + \lambda_s \beta E_t \left( \Pi_{s,t+1}^X \right)^{\theta_s} \tilde{K}_{s,t+1}^X - \tilde{K}_{s,t}^X \Bigg\} \\
& + \sum_{(s,u)} \Xi_{7t}^{(s,u)} \left\{ \lambda_s \left( \frac{P_{st}^u}{P_{st-1}^u} \right)^{\theta_s} \Delta_{s,t-1}^u + (1-\lambda_s) \left( f_s \left( \frac{P_{s,t}^u}{P_t}, \Pi_t; \frac{P_{s,t-1}^u}{P_{t-1}} \right) \right)^{\theta_s} - \Delta_{st}^u \right\} \\
& + \Xi_{8t} \left\{ \xi C^* Q_t^{\frac{1}{\sigma}} - C_t \right\} \Bigg] + \Lambda D_0 \\
& + \Xi_9 \left\{ \sum_{t=0}^{\infty} \left[ \beta^t \sum_{s \in S} \left( \left( \frac{P_{st}^X/P_t}{Q_t P_s^*} \right)^{-\theta_s^*} X_s^* \frac{P_{st}^X}{Q_t P_t} - \frac{\alpha_{sm}}{\alpha_{sl}} \frac{W_t/P_t}{Q_t} L_{st} \right) \right] - D_0 \right\}.
\end{aligned}
$$

By taking the first-order condition with respect to $C_t$ , $Q_t$, $L_{st}$, $W_t/P_t$, $\Delta_{st}$, $P_{st}/P_t$, $P_{Xt}^X/P_t$, $\Pi_t$, $\tilde{F}_{s,t}^u$, $\tilde{K}_{s,t}^u$, it can be shown that there exists a solution to this system of first-order conditions that satisfies $\Pi_t = \Pi_{s,t}^u = 1$, $\Delta_t^{(s,t)} = 1$, $C_t = \bar{C}$, $L_t = \bar{L}$,$Q_t = \bar{Q}$, $W_t/P_t = \bar{W}$, $P_{st}^u/P_t = \bar{P}_s^u$, $\tilde{F}_{s,t}^u = \bar{F}_s^u$ and $\tilde{K}_{s,t}^u = \bar{K}_s^u$ with constant Lagrange multipliers. To do this, use the following relationships: $f_s(P_s, 1; P_s) = 1$, $f_{s1}(P_s, 1; P_s) = \frac{-\lambda_s}{1-\lambda_s} P_s^{-1}$, $f_{s2}(P_s, 1; P_s) = \frac{-\lambda_s}{1-\lambda_s}$ and $f_{s3}(P_s, 1; P_s) = \frac{\lambda_s}{1-\lambda_s} P_s^{-1}$ to see that the first-order conditions reduce to 10 linear equations with respect to $\left(\Xi_{1t}, \Xi_{2t}, \Xi_{3t}, \Xi_{6t}^M, \Xi_{6t}^X, \Xi_{6t}^{XX}, \Xi_{7t}^M, \Xi_{7t}^X, \Xi_{7t}^{XX}, \Xi_{8t}\right)$. Thus, generically, we can solve the system given $C$, $\{L_s\}$,$Q$, $W$, $P_s^u$, and $F_s^u$ . The values for $C$, $\{L_s\}$,$Q$, $W$, $P_s^u$, $F_s^u$ are the solutions to the constraints with zero inflation. Thus we have shown that the optimal steady state is characterized by the following.

$$\begin{cases}\psi_s C = P_s C_s \\ \frac{L^{\phi}}{C^{-\sigma}} = W \\ C = \xi C^* Q^{\frac{1}{\sigma}} \\ \chi_s^{-1}\left(\frac{QQ_s^*}{\alpha_{sm}}\right)^{\alpha_{sm}}\left(\frac{W}{\alpha_{sl}}\right)^{\alpha_{sl}}\frac{1}{Z_s} = P_s \\ \chi_s^{-1}\nu_s^{-1}\frac{\theta_s^*}{\theta_s^*-1}\left(\frac{QQ_s^*}{\alpha_{sm}}\right)^{\alpha_{sm}}\left(\frac{W}{\alpha_{sl}}\right)^{\alpha_{sl}}\frac{1}{Z_s} = P_s^X \\ Z_s L_s = \left(\frac{\alpha_{sl}}{\alpha_{sm}}\frac{QQ_s^*}{W}\right)^{\alpha_{sm}}\left(C_s + \left(\frac{P_s^X}{QP_s^*}\right)^{-\theta_s^*} X_s^*\right) \\ M_s = \frac{\alpha_{sm}}{\alpha_{sl}}\frac{W}{QQ_s^*}L_s = \left(\frac{W}{\alpha_{sl}}\right)^{\alpha_{sl}}\left(\frac{QQ_s^*}{\alpha_{sm}}\right)^{\alpha_{sm}-1}(C_s + X_s)\frac{1}{Z_s} = \alpha_{sm}\chi_s\frac{P_s}{QQ_s^*}(C_s + X_s) \\ \sum_{t=0}^{\infty}\left[\beta^t \sum_{s\in S}\left(\left(\frac{P_s^X}{QP_s^*}\right)^{-\theta_s^*} X_s^* \frac{P_s^X}{Q} - M_s Q_s^*\right)\right] = D_0\end{cases}$$

### B.4.2 Properties

Note that by the definition of $\xi := \left(\frac{\Lambda}{\left(C_0^*\right)^{-\sigma}/P_0^*}\right)^{-\frac{1}{\sigma}}$ and the assumption that $P^* = 1$, we have

$$C = \xi C^* Q^{\frac{1}{\sigma}} \Leftrightarrow C^{\sigma}\frac{\Lambda}{\left(C_0^*\right)^{-\sigma}/P_0^*} = C^{*\sigma} Q \Leftrightarrow \frac{\Lambda}{Q} = C^{-\sigma}.$$

Note also that by the definition of $\mu_s, \xi_s$,

$$\mu_s = \frac{M_s Q_s^*}{\sum_{s\in S}\left(\left(\frac{P_s^X}{QP_s^*}\right)^{-\theta_s^*} X_s^* \frac{P_s^X}{Q} - M_s Q_s^*\right)}, \xi_s = \frac{\left(\frac{P_s^X}{QP_s^*}\right)^{-\theta_s^*} X_s^* \frac{P_s^X}{Q}}{\sum_{s\in S}\left(\left(\frac{P_s^X}{QP_s^*}\right)^{-\theta_s^*} X_s^* \frac{P_s^X}{Q} - M_s Q_s^*\right)}.$$

Thus,

$$\mu_s (1-\beta) D_0 = M_s Q_s^*, \xi_s (1-\beta) D_0 = \left(\frac{P_s^X}{QP_s^*}\right)^{-\theta_s^*} X_s^* \frac{P_s^X}{Q}.$$

Let us first show the following relationships since these appear a few times.

$$\begin{cases}\Lambda\bar{D}_0 (1-\beta)\xi_s(1-\theta_s^*) = -\chi_s^{-1}\nu_s^{-1}\theta_s^*\phi_{sx}\frac{\phi_{ls}}{\alpha_{sl}}L^{1+\phi} \\ C^{1-\sigma} = \sum_{s\in S}\chi_s^{-1}\phi_{sc}\frac{\phi_{ls}}{\alpha_{sl}}L^{1+\phi} \\ \Lambda\bar{D}_0(1-\beta)\mu_s = \alpha_{sm}\frac{\phi_{ls}}{\alpha_{sl}}L^{1+\phi} \\ M_w M_l^{-1} = L^{1+\phi}\boldsymbol{\phi}_l' d\left(\boldsymbol{\alpha}_l\right)^{-1} \\ \psi_s = \frac{\chi_s^{-1}\phi_{sc}\frac{\phi_{ls}}{\alpha_{sl}}}{\sum_{s'\in S}\chi_{s'}^{-1}\phi_{s'c}\frac{\phi_{ls'}}{\alpha_{s'l}}}\end{cases}$$

First,

$$\Lambda\bar{D}_0(1-\beta)\xi_s(1-\theta_s^*) = -\chi_s^{-1}\nu_s^{-1}\theta_s^*\phi_{sx}\frac{\phi_{ls}}{\alpha_{sl}}L^{1+\phi}$$

where I used the characterization of the steady state and

$$Z_s L_s = \left(\frac{\alpha_{sl}}{\alpha_{sm}} \frac{QQ_s^*}{W}\right)^{\alpha_{sm}} (C_s + X_s) \Rightarrow \left(\frac{\alpha_{sl}}{\alpha_{sm}} \frac{QQ_s^*}{W}\right)^{\alpha_{sm}} = \frac{Z_s L_s}{C_s + X_s}.$$

Next, $C^{1-\sigma}$ becomes

$$C^{1-\sigma} = \boldsymbol{\phi}_l' diag \boldsymbol{\chi}^{-1} diag \boldsymbol{\alpha}_l^{-1} \boldsymbol{\phi}_c L^{1+\phi},$$

where I again used the relationship derived from the resource constraint. Finally,$\Lambda \bar{D}_0 (1-\beta) \mu_s$ can be calculated as follows.

$$\Lambda \bar{D}_0 (1-\beta) \mu_s = L^{1+\phi} \alpha_{sm} \chi_s \frac{\psi_s}{\phi_{sc}} \boldsymbol{\phi}_l' diag \boldsymbol{\chi}^{-1} diag \boldsymbol{\alpha}_l^{-1} \boldsymbol{\phi}_c$$

Thus, recall that

$$\begin{aligned} \phi_{ls} &= \frac{L_s}{L} \\ &= \left(\sum_{s' \in S} \chi_{s'}^{-1} \phi_{sc'} \frac{\phi_{ls'}}{\alpha_{s'l}}\right) \frac{\psi_s}{\phi_{sc}} \alpha_{sl} \chi_s. \end{aligned}$$

Therefore, we have

$$\psi_s = \frac{\chi_s^{-1} \phi_{sc} \frac{\phi_{ls}}{\alpha_{sl}}}{\sum_{s' \in S} \chi_{s'}^{-1} \phi_{sc'} \frac{\phi_{ls'}}{\alpha_{s'l}}}.$$

## B.5 Second-order approximated welfare function

**Exact relationships** In the following, I will use the following equilibrium relationships.

$$\begin{cases} \psi_s C_t = \frac{P_{st}}{P_t} C_{st} \\ \frac{L_t^{\phi}}{C_t^{-\sigma}} = \frac{W_t}{P_t} \\ C_t = \xi C_t^* Q_t^{\frac{1}{\sigma}} \\ M_{st} = \frac{\alpha_{sm}}{\alpha_{sl}} \frac{W_t/P_t}{Q_t Q_{st}^*/P_t^*} L_{st} \\ Y_{s,t} = C_{st}, \ Y_{s,t}^X = \left(\frac{P_{st}^X/P_t}{Q_t P_{st}^*/P_t^*}\right)^{-\theta_s^*} X_{st}^* = X_{st} \\ C_t = \prod_{s \in S} C_{st}^{\psi_s} \end{cases}$$

$$\Rightarrow \begin{cases} p_{st} = c_t - c_{st} = \sum_{s' \in S} \psi_{s'} c_{s't} - c_{st} & (a) \\ w_t = \phi l_t + \sigma c_t = \phi l_t + \sigma \sum_{s' \in S} \psi_{s'} c_{s't} & (b) \\ q_t = \sigma (c_t - c_t^*) = \sigma \left(\sum_{s' \in S} \psi_{s'} c_{s't} - c_t^*\right) & (c) \\ m_{st} = w_t - q_t - q_{st}^* + l_{st} = \phi l_t + l_{st} + \sigma c_t^* - q_{st}^* & (d) \\ y_{st} = c_{st}, \ y_{st}^X = -\theta_s^* \left(p_{st}^X - q_t - p_{st}^*\right) + x_{st}^* = x_{st} & (e) \\ c_t = \sum_{s \in S} \psi_s c_{st} & (f) \end{cases} \tag{32}$$

One can see that the $\left\{p_{st}, p_{st}^X, m_{st}, y_{st}, y_{st}^X\right\}_{s\in S}, c_t, w_t, q_t$ can be written as linear functions of $\{c_{st}, x_{st}, l_{st}\}_{s\in S}$ and $l_t$. The rest of the equations are the resource constraint

$$Z_{st}L_{st} = \left(\frac{\alpha_{sl}}{\alpha_{sm}}\frac{Q_t Q_{st}^*/P_t^*}{W_t/P_t}\right)^{\alpha_{sm}} \left\{\Delta_{st}C_{st} + \Delta_{st}^X \left(\frac{P_{st}^X/P_t}{Q_t P_{st}^*/P_t^*}\right)^{-\theta_s^*} X_{st}^*\right\}$$

and the initial level of debt

$$E_0 \sum_{t=0}^{\infty} \left[\beta^t \frac{(C_t^*)^{-\sigma}}{(C_0^*)^{-\sigma}/P_0^*} \sum_{s\in S} \left(\left(\frac{P_{st}^X/P_t}{Q_t P_{st}^*/P_t^*}\right)^{-\theta_s^*} X_{st}^* \frac{P_{st}^X}{Q_t P_t} - M_{st}\frac{Q_{st}^*}{P_t^*}\right)\right] = D_0.$$

**Naive Welfare** Since welfare is

$$\mathcal{W} = E_0 \sum_{t=0}^{\infty} \beta^t \left[\underbrace{\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\phi}}{1+\phi}}_{=:U_t}\right] + \Lambda D_0,$$

denote the steady-state value of the welfare by

$$\bar{\mathcal{W}} = \frac{1}{1-\beta}U + \Lambda \bar{D}_0.$$

Subtracting this from welfare can still serve as our welfare criterion.

$$\mathcal{W} - \bar{\mathcal{W}} = E_0 \sum_{t=0}^{\infty} \beta^t \left[U_t - U\right] + \Lambda \bar{D}_0 \left(\frac{D_0 - \bar{D}_0}{\bar{D}_0}\right),$$

The second-order Taylor expansion of $U_t(C_t, L_t) = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\phi}}{1+\phi}$ around the steady state $(C, L)$ is

$$U_t - U \approx C^{1-\sigma}\left(c_t + \frac{1-\sigma}{2}c_t^2\right) - L^{1+\phi}\left(l_t + \frac{1+\phi}{2}l_t^2\right)$$

Using $L_t = \sum_{s\in S} L_{st}$,

$$l_t + \frac{1}{2}l_t^2 = \sum_{s\in S}\phi_{ls}l_{st} + \frac{1}{2}\sum_{s\in S}\phi_{ls}l_{st}^2$$

where

$$\phi_{ls} = \frac{L_s}{L}.$$

Plugging this into the above,

$$\begin{aligned} U_t - U &\approx C^{1-\sigma}\left(c_t + \frac{1-\sigma}{2}c_t^2\right) - L^{1+\phi}\left(\sum_{s\in S}\phi_{ls}\hat{l}_{st} + \frac{1}{2}\sum_{s\in S}\phi_{ls}\hat{l}_{st}^2 + \frac{\phi}{2}\hat{l}_t^2\right) \\ &= C^{1-\sigma}c_t - L^{1+\phi}\sum_{s\in S}\phi_{ls}l_{st} + \frac{1}{2}S_{Wt} \end{aligned}$$

where

$$S_{Wt} = C^{1-\sigma}(1-\sigma)c_t^2 - L^{1+\phi}\left(\sum_{s\in S}\phi_{ls}l_{st}^2 + \phi l_t^2\right).$$

Similarly to the standard closed economy NK models, we can use the approximated resource constraint to derive the relationship between $l_t$ and $c_t$. First, take the second-order approximation as follows. Since $\Delta_{st}$ is of second order or higher,

$$Z_{st}L_{st} = \left(\frac{\alpha_{sl}}{\alpha_{sm}}\frac{Q_tQ_{st}^*/P_t^*}{W_t/P_t}\right)^{\alpha_{sm}}\left\{\Delta_{st}C_{st} + \Delta_{st}^X\left(\frac{P_{st}^X/P_t}{Q_tP_{st}^*/P_t^*}\right)^{-\theta_s^*}X_{st}^*\right\}$$

$$\begin{aligned}&\Rightarrow z_{st} + l_{st} - \alpha_{sm}(q_t + q_{st}^* - w_t) + \frac{1}{2}\left\{z_{st} + l_{st} - \alpha_{sm}(q_t + q_{st}^* - w_t)\right\}^2\\ &\quad = \phi_{sc}\left(\Delta_{st} + c_{st} + \frac{1}{2}c_{st}^2\right) + \phi_{sx}\left(\Delta_{st}^X + x_{st} + \frac{1}{2}x_{st}^2\right),\end{aligned}$$

where

$$\phi_{sc} = \frac{C_s}{C_s + X_s},\ \phi_{sx} = \frac{X_s}{C_s + X_s}.$$

Utilize equation 32-(b),(c), and

$$l_t + \frac{1}{2}l_t^2 = \sum_{s\in S}\phi_{ls}l_{st} + \frac{1}{2}\sum_{s\in S}\phi_{ls}l_{st}^2,$$

$$\begin{aligned}&z_{st} + l_{st} - \alpha_{sm}\left(-\sigma c_t^* + q_{st}^* - \phi\left(\sum_{s'\in S}\phi_{ls'}l_{s't} + \frac{1}{2}\sum_{s'\in S}\phi_{ls'}l_{s't}^2 - \frac{1}{2}l_t^2\right)\right)\\ &\quad + \frac{1}{2}\left\{z_{st} + l_{st} - \alpha_{sm}(-\sigma c_t^* + q_{st}^* - \phi l_t)\right\}^2\\ &= \phi_{sc}\left(\Delta_{st} + c_{st} + \frac{1}{2}c_{st}^2\right) + \phi_{sx}\left(\Delta_{st}^X + x_{st} + \frac{1}{2}x_{st}^2\right),\end{aligned}$$

Solving for the linear term in $l_{st}$, and gathering the quadratic terms together,

$$\begin{aligned}&l_{st} + \alpha_{sm}\phi\left(\sum_{s'\in S}\phi_{ls'}l_{s't}\right)\\ &= \phi_{sc}c_{st} + \phi_{sx}x_{st} + \alpha_{sm}(-\sigma c_t^* + q_{st}^*) - \alpha_{sm}\phi\left(\frac{1}{2}\sum_{s'\in S}\phi_{ls'}l_{s't}^2 - \frac{1}{2}l_t^2\right)\\ &\quad + \phi_{sc}\left(\Delta_{st} + \frac{1}{2}c_{st}^2\right) + \phi_{sx}\left(\Delta_{st}^X + \frac{1}{2}x_{st}^2\right) - z_{st} - \frac{1}{2}\left\{z_{st} + l_{st} - \alpha_{sm}(-\sigma c_t^* + q_{st}^* - \phi l_t)\right\}^2\end{aligned}$$

In matrix,

$$\begin{aligned}&\underbrace{\left[I + \phi d(\boldsymbol{\alpha}_m)\mathbf{1}_{S\times 1}\boldsymbol{\phi}_l'\right]}_{=:M_l}\boldsymbol{l}_t\\ &= d(\boldsymbol{\phi}_c)\boldsymbol{c}_t + d(\boldsymbol{\phi}_x)\boldsymbol{x}_t + d(\boldsymbol{\alpha}_m)(-\mathbf{1}_{S\times 1}\sigma c_t^* + \boldsymbol{q}_t^*) - \boldsymbol{z}_t\\ &\quad - \frac{1}{2}\boldsymbol{\alpha}_m\phi\boldsymbol{l}_t'(d(\boldsymbol{\phi}_l) - \boldsymbol{\phi}_l\boldsymbol{\phi}_l')\boldsymbol{l}_t + \frac{1}{2}d(\boldsymbol{\phi}_c)(2\boldsymbol{\Delta}_t + d(\boldsymbol{c}_t)\boldsymbol{c}_t) + \frac{1}{2}d(\boldsymbol{\phi}_x)\left(2\boldsymbol{\Delta}_t^X + d(\boldsymbol{x}_t)\boldsymbol{x}_t\right)\\ &\quad - \frac{1}{2}d(d(\boldsymbol{\phi}_c)\boldsymbol{c}_t + d(\boldsymbol{\phi}_x)\boldsymbol{x}_t)(d(\boldsymbol{\phi}_c)\boldsymbol{c}_t + d(\boldsymbol{\phi}_x)\boldsymbol{x}_t)\end{aligned}$$

Thus, up to first order,

$$\boldsymbol{l}_t = M_l^{-1}\left\{d\left(\boldsymbol{\phi}_c\right)\boldsymbol{c}_t + d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t + d\left(\boldsymbol{\alpha}_m\right)\left(-\mathbf{1}_{S\times 1}\sigma c_t^* + \boldsymbol{q}_t^*\right) - \boldsymbol{z}_t\right\}.$$

Furthermore, noticing that $\sum_{t=0}^{\infty}\beta^t E_0 \Delta_{st}^u = \sum_{t=0}^{\infty}\beta^t E_0 \frac{\theta_s}{2\kappa_s}\left(\pi_{s,t}^u\right)^2$ ,where $\kappa_s = \frac{(1-\lambda_s)(1-\beta\lambda_s)}{\lambda_s}$ the infinite sum becomes

$$\sum_{t=0}^{\infty}\beta^t E_0 \hat{\boldsymbol{l}}_t = M_l^{-1}\sum_{t=0}^{\infty}\beta^t E_0\left[d\left(\boldsymbol{\phi}_c\right)\boldsymbol{c}_t + d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t\right] + \frac{1}{2}\sum_{t=0}^{\infty}\beta^t E_0 \boldsymbol{S}_{Rt} + t.i.p.,$$

where

$$\begin{aligned}\boldsymbol{S}_{Rt} &= M_l^{-1}\left[d\left(\boldsymbol{\phi}_c\right)d\left(\boldsymbol{c}_t\right)\boldsymbol{c}_t + d\left(\boldsymbol{\phi}_x\right)d\left(\boldsymbol{x}_t\right)\boldsymbol{x}_t - \boldsymbol{\alpha}_m\phi\boldsymbol{l}_t'\left(d\left(\boldsymbol{\phi}_l\right) - \boldsymbol{\phi}_l\boldsymbol{\phi}_l'\right)\boldsymbol{l}_t\right] \\ &- M_l^{-1}\left[d\left(d\left(\boldsymbol{\phi}_c\right)\boldsymbol{c}_t + d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t\right)\left(d\left(\boldsymbol{\phi}_c\right)\boldsymbol{c}_t + d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t\right)\right] \\ &+ M_l^{-1}\left[d\left(\boldsymbol{\theta}\right)d\left(\boldsymbol{\kappa}\right)^{-1}\left\{d\left(\boldsymbol{\phi}_c\right)d\left(\boldsymbol{\pi}_t\right)\boldsymbol{\pi}_t + d\left(\boldsymbol{\phi}_x\right)d\left(\boldsymbol{\pi}_t^X\right)\boldsymbol{\pi}_t^X\right\}\right]\end{aligned}$$

By approximating the lifetime international budget condition, we can approximate the initial debt $\frac{D_0-\bar{D}_0}{\bar{D}_0} \approx \hat{d}_0 + \frac{1}{2}\hat{d}_0^2$ as

$$d_0 + \frac{1}{2}d_0^2 = (1-\beta)E_0\sum_{t=0}^{\infty}\beta^t\left[\sum_{s\in S}\xi_s\frac{\theta_s^*-1}{\theta_s^*}x_{st} - \boldsymbol{\mu}'M_m\hat{\boldsymbol{l}}_t + \frac{1}{2}\frac{S_{Dt}}{1-\beta}\right] + t.i.p.$$

$$\begin{aligned}\frac{\tilde{S}_{Dt}}{1-\beta} :&= \boldsymbol{x}_t'd\left(\boldsymbol{\theta}^*-1\right)d\left(\boldsymbol{\theta}^*\right)^{-1}d\left(\boldsymbol{\xi}\right)d\left(\boldsymbol{\theta}^*-1\right)d\left(\boldsymbol{\theta}^*\right)^{-1}\boldsymbol{x}_t \\ &+ 2\left(-\sigma\hat{c}_t^*\mathbf{1}_{S\times 1} + d\left(\boldsymbol{\theta}^*\right)^{-1}\boldsymbol{x}_t^* + \boldsymbol{p}_t^*\right)'d\left(\boldsymbol{\xi}\right)d\left(\boldsymbol{\theta}^*-1\right)d\left(\boldsymbol{\theta}^*\right)^{-1}\boldsymbol{x}_t \\ &- \boldsymbol{l}_t'M_m'd\left(\boldsymbol{\mu}\right)M_m\boldsymbol{l}_t\end{aligned}$$

$$\begin{aligned}\frac{S_{Dt}}{1-\beta} &= \frac{\tilde{S}_{Dt}}{1-\beta} - \phi\boldsymbol{\mu}'\mathbf{1}_{S\times 1}\boldsymbol{l}_t'\left(d\left(\boldsymbol{\phi}_l\right) - \boldsymbol{\phi}_l\boldsymbol{\phi}_l'\right)\boldsymbol{l}_t \\ &= \boldsymbol{x}_t'd\left(\boldsymbol{\theta}^*-1\right)d\left(\boldsymbol{\theta}^*\right)^{-1}d\left(\boldsymbol{\xi}\right)d\left(\boldsymbol{\theta}^*-1\right)d\left(\boldsymbol{\theta}^*\right)^{-1}\boldsymbol{x}_t \\ &+ 2\left(-\sigma\hat{c}_t^*\mathbf{1}_{S\times 1} + d\left(\boldsymbol{\theta}^*\right)^{-1}\boldsymbol{x}_t^* + \boldsymbol{p}_t^*\right)'d\left(\boldsymbol{\xi}\right)d\left(\boldsymbol{\theta}^*-1\right)d\left(\boldsymbol{\theta}^*\right)^{-1}\boldsymbol{x}_t \\ &- \boldsymbol{l}_t'\left(M_m'd\left(\boldsymbol{\mu}\right)M_m + \phi\boldsymbol{\mu}'\mathbf{1}_{S\times 1}\left(d\left(\boldsymbol{\phi}_l\right) - \boldsymbol{\phi}_l\boldsymbol{\phi}_l'\right)\right)\boldsymbol{l}_t\end{aligned}$$

$$M_m = \phi\mathbf{1}_{S\times 1}\boldsymbol{\phi}_l' + I$$

Plugging the expressions for $U_t - U$, $\sum_{t=0}^{\infty}\beta^t E_0\left[\hat{l}_t\right]$, and $\frac{D_0-\bar{D}_0}{\bar{D}_0}$ obtained above into the equation for $\mathcal{W} - \bar{\mathcal{W}}$, we obtain the following welfare criterion

$$\begin{aligned}U_t - U &\approx C^{1-\sigma}\left(c_t + \frac{1-\sigma}{2}c_t^2\right) - L^{1+\phi}\left(\sum_{s\in S}\phi_{ls}\hat{l}_{st} + \frac{1}{2}\sum_{s\in S}\phi_{ls}\hat{l}_{st}^2 + \frac{\phi}{2}\hat{l}_t^2\right) \\ &= C^{1-\sigma}c_t - L^{1+\phi}\sum_{s\in S}\phi_{ls}l_{st} + \frac{1}{2}S_{Wt}\end{aligned}$$

$$
\begin{aligned}
\mathcal{W}-\bar{\mathcal{W}} &= \sum_{t=0}^{\infty}\beta^t E_0\left[C^{1-\sigma}c_t\right]-L^{1+\phi}\boldsymbol{\phi}_l'\sum_{t=0}^{\infty}\beta^t E_0\boldsymbol{l}_t \\
&+\Lambda\bar{D}_0\left(1-\beta\right)E_0\sum_{t=0}^{\infty}\beta^t\left[\boldsymbol{\xi}'d\left(\boldsymbol{\theta}^*-1\right)d\left(\boldsymbol{\theta}^*\right)^{-1}\boldsymbol{x}_t-\boldsymbol{\mu}'M_m\boldsymbol{l}_t\right] \\
&+\frac{1}{2}\sum_{t=0}^{\infty}\beta^t E_0\left[S_{Wt}+\Lambda\bar{D}_0 S_{Dt}\right]+t.i.p. \\
&=\sum_{t=0}^{\infty}\beta^t E_0\underbrace{\left[C^{1-\sigma}\boldsymbol{\psi}'-M_w M_l^{-1}d\left(\boldsymbol{\phi}_c\right)\right]}_{=:L^{1+\phi}f_c(\chi,\nu)}\boldsymbol{c}_t \\
&+\sum_{t=0}^{\infty}\beta^t E_0\underbrace{\left[\Lambda\bar{D}_0\left(1-\beta\right)\boldsymbol{\xi}'d\left(\boldsymbol{\theta}^*-1\right)d\left(\boldsymbol{\theta}^*\right)^{-1}-M_w M_l^{-1}d\left(\boldsymbol{\phi}_x\right)\right]}_{=:L^{1+\phi}f_x(\chi,\nu)}\boldsymbol{x}_t \\
&+\frac{1}{2}\sum_{t=0}^{\infty}\beta^t E_0\left[S_{Wt}+\Lambda\bar{D}_0 S_{Dt}-M_w\boldsymbol{S}_{Rt}\right]+t.i.p.
\end{aligned}
$$

Finally, I show that $f_c(\chi,\nu)$ and $f_x(\chi,\nu)$ can be simplified as

$$
f_c(\chi,\nu)=\boldsymbol{\phi}_l'd\left(\boldsymbol{\alpha}_l\right)^{-1}d\left(\boldsymbol{\phi}_c\right)\left(d\left(\boldsymbol{\chi}\right)^{-1}-I\right)
$$

$$
f_x(\chi,\nu)=\boldsymbol{\phi}_l'd\left(\boldsymbol{\alpha}_l\right)^{-1}d\left(\boldsymbol{\phi}_x\right)\left(d\left(\boldsymbol{\chi}\right)^{-1}d\left(\boldsymbol{\nu}\right)^{-1}-I\right).
$$

To see this, first note the following.

$$
\begin{aligned}
M_w M_l^{-1} &= \left(L^{1+\phi}\boldsymbol{\phi}_l'+\Lambda\bar{D}_0\left(1-\beta\right)\boldsymbol{\mu}'\left[I+\phi\mathbf{1}_{S\times1}\boldsymbol{\phi}_l'\right]\right)\left[I+\phi d\left(\boldsymbol{\alpha}_m\right)\mathbf{1}_{S\times1}\boldsymbol{\phi}_l'\right]^{-1} \\
&= L^{1+\phi}\boldsymbol{\phi}_l'd\left(\boldsymbol{\alpha}_l\right)^{-1}
\end{aligned}
$$

Using the properties derived in Appendix B.4, the desired relationships hold as follows;

$$
\begin{aligned}
f_c(\chi,\nu) &= \frac{C^{1-\sigma}}{L^{1+\phi}}\boldsymbol{\psi}'-\frac{1}{L^{1+\phi}}M_w M_l^{-1}d\left(\boldsymbol{\phi}_c\right) \\
&= \sum_{s\in S}\chi_s^{-1}\phi_{sc}\frac{\phi_{ls}}{\alpha_{sl}}\boldsymbol{\psi}'-\boldsymbol{\phi}_l'd\left(\boldsymbol{\alpha}_l\right)^{-1}d\left(\boldsymbol{\phi}_c\right) \\
&= \boldsymbol{\phi}_l'd\left(\boldsymbol{\alpha}_l\right)^{-1}d\left(\boldsymbol{\phi}_c\right)\left(d\left(\boldsymbol{\chi}\right)^{-1}-I\right)
\end{aligned}
$$

$$
\begin{aligned}
f_x(\chi,\nu) &= \frac{1}{L^{1+\phi}}\Lambda\bar{D}_0\left(1-\beta\right)\boldsymbol{\xi}'d\left(\boldsymbol{\theta}^*-1\right)d\left(\boldsymbol{\theta}^*\right)^{-1}-\frac{1}{L^{1+\phi}}M_w M_l^{-1}d\left(\boldsymbol{\phi}_x\right) \\
&= \boldsymbol{\phi}_l'd\left(\boldsymbol{\alpha}_l\right)^{-1}d\left(\boldsymbol{\phi}_x\right)\left(d\left(\boldsymbol{\chi}\right)^{-1}d\left(\boldsymbol{\nu}\right)^{-1}-I\right).
\end{aligned}
$$

## B.6 Natural rate under the efficient steady state

When the steady state is efficient, $\chi_M = \chi_X = \chi_T = 1$, and all the $f$ are zeros. Thus, recalling $M_w M_l^{-1} = L^{1+\phi}\boldsymbol{\phi}_l' diag\left(\boldsymbol{\alpha}_l\right)^{-1}, M_m = \phi \mathbf{1}_{S\times 1}\boldsymbol{\phi}_l' + I$,

$$
\begin{aligned}
& L^{-(1+\phi)}\left(\mathcal{W} - \bar{\mathcal{W}}\right)|_{\text{efficient}} \\
=& \frac{L^{-(1+\phi)}}{2}\sum_{t=0}^{\infty}\beta^t E_0\left[S_{Wt} - M_w \boldsymbol{S}_{Rt} + \Lambda \bar{D}_0 S_{Dt}\right] + t.i.p. \\
=& \frac{1}{2}\sum_{t=0}^{\infty}\beta^t E_0\Big[\left(1-\sigma\right)\boldsymbol{\phi}_l' d\left(\boldsymbol{\chi}\right)^{-1} d\left(\boldsymbol{\alpha}_l\right)^{-1}\boldsymbol{\phi}_c \boldsymbol{c}_t' \boldsymbol{\psi}\boldsymbol{\psi}' \boldsymbol{c}_t - \boldsymbol{\phi}_l' d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(\boldsymbol{\phi}_c\right) d\left(\boldsymbol{c}_t\right)\boldsymbol{c}_t \\
& - \boldsymbol{\phi}_l' d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(\boldsymbol{\phi}_x\right) d\left(\boldsymbol{x}_t\right)\boldsymbol{x}_t + \boldsymbol{x}_t' d\left(\boldsymbol{\theta}^* - 1\right) d\left(\boldsymbol{\theta}^*\right)^{-1} d\left(\boldsymbol{\phi}_l\right) d\left(\boldsymbol{\chi}\right)^{-1} d\left(\boldsymbol{\nu}\right)^{-1} d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t \\
& + \boldsymbol{\phi}_l' d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(d\left(\boldsymbol{\phi}_c\right)\boldsymbol{c}_t + d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t\right)\left(d\left(\boldsymbol{\phi}_c\right)\boldsymbol{c}_t + d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t\right) \\
& - \boldsymbol{l}_t'\left(d\left(\boldsymbol{\phi}_l\right) + \phi\boldsymbol{\phi}_l\boldsymbol{\phi}_l'\right)\boldsymbol{l}_t + \boldsymbol{\phi}_l' d\left(\boldsymbol{\alpha}_l\right)^{-1}\boldsymbol{\alpha}_m \phi \boldsymbol{l}_t'\left(d\left(\boldsymbol{\phi}_l\right) - \boldsymbol{\phi}_l\boldsymbol{\phi}_l'\right)\boldsymbol{l}_t \\
& - \boldsymbol{l}_t'\left(M_m' d\left(\boldsymbol{\phi}_l\right) d\left(\boldsymbol{\alpha}_m\right) d\left(\boldsymbol{\alpha}_l\right)^{-1} M_m + \phi \mathbf{1}_{1\times S} d\left(\boldsymbol{\phi}_l\right) d\left(\boldsymbol{\alpha}_m\right) d\left(\boldsymbol{\alpha}_l\right)^{-1}\mathbf{1}_{S\times 1}\left(d\left(\boldsymbol{\phi}_l\right) - \boldsymbol{\phi}_l\boldsymbol{\phi}_l'\right)\right)\boldsymbol{l}_t \\
& + 2\left(-\sigma\hat{c}_t^*\mathbf{1}_{S\times 1} + d\left(\boldsymbol{\theta}^*\right)^{-1}\boldsymbol{x}_t^* + \boldsymbol{p}_t^*\right)' d\left(\boldsymbol{\phi}_l\right) d\left(\boldsymbol{\chi}\right)^{-1} d\left(\boldsymbol{\nu}\right)^{-1} d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t \\
& - \boldsymbol{\phi}_l' d\left(\boldsymbol{\alpha}_l\right)^{-1}\left[d\left(\boldsymbol{\theta}\right) d\left(\boldsymbol{\kappa}\right)^{-1}\left\{d\left(\boldsymbol{\phi}_c\right) d\left(\boldsymbol{\pi}_t\right)\boldsymbol{\pi}_t + d\left(\boldsymbol{\phi}_x\right) d\left(\boldsymbol{\pi}_t^X\right)\boldsymbol{\pi}_t^X\right\}\right] + t.i.p.
\end{aligned}
$$

where

$$
\boldsymbol{l}_t = M_l^{-1}\left\{d\left(\boldsymbol{\phi}_c\right)\boldsymbol{c}_t + d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t + d\left(\boldsymbol{\alpha}_m\right)\left(-\mathbf{1}_{S\times 1}\sigma c_t^* + \boldsymbol{q}_t^*\right) - \boldsymbol{z}_t\right\}.
$$

Collecting terms, and recalling $\boldsymbol{\chi} = \boldsymbol{\nu} = \mathbf{1}_{S\times 1}$,

$$
\begin{aligned}
& L^{-(1+\phi)}\left(\mathcal{W} - \bar{\mathcal{W}}\right)|_{\text{efficient}} \\
=& \frac{1}{2}\sum_{t=0}^{\infty}\beta^t E_0\Big[\boldsymbol{c}_t'\left(\left(1-\sigma\right)\boldsymbol{\phi}_l' d\left(\boldsymbol{\alpha}_l\right)^{-1}\boldsymbol{\phi}_c\boldsymbol{\psi}\boldsymbol{\psi}'\right)\boldsymbol{c}_t \\
& - \boldsymbol{c}_t' d\left(\boldsymbol{\phi}_x\right) d\left(\boldsymbol{\phi}_l\right) d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(\boldsymbol{\phi}_c\right)\boldsymbol{c}_t \\
& - \boldsymbol{x}_t'\left(d\left(\boldsymbol{\theta}^*\right)^{-1} - d\left(\boldsymbol{\phi}_x\right)\right) d\left(\boldsymbol{\phi}_l\right) d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t \\
& + 2\boldsymbol{c}_t' d\left(\boldsymbol{\phi}_c\right) d\left(\boldsymbol{\phi}_l\right) d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t \\
& - \boldsymbol{l}_t' M_l' d\left(\boldsymbol{\alpha}_l\right)^{-1}\left(d\left(\boldsymbol{\phi}_l\right) + \phi\boldsymbol{\phi}_l\boldsymbol{\phi}_l'\right)\boldsymbol{l}_t \\
& + 2\left(-\sigma\hat{c}_t^*\mathbf{1}_{S\times 1} + d\left(\boldsymbol{\theta}^*\right)^{-1}\boldsymbol{x}_t^* + \boldsymbol{p}_t^*\right)' d\left(\boldsymbol{\phi}_l\right) d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t \\
& - \boldsymbol{\phi}_l' d\left(\boldsymbol{\alpha}_l\right)^{-1}\left[d\left(\boldsymbol{\theta}\right) d\left(\boldsymbol{\kappa}\right)^{-1}\left\{d\left(\boldsymbol{\phi}_c\right) d\left(\boldsymbol{\pi}_t\right)\boldsymbol{\pi}_t + d\left(\boldsymbol{\phi}_x\right) d\left(\boldsymbol{\pi}_t^X\right)\boldsymbol{\pi}_t^X\right\}\right] + t.i.p.
\end{aligned}
$$

Using

$$
\boldsymbol{l}_t = M_l^{-1}\left\{d\left(\boldsymbol{\phi}_c\right)\boldsymbol{c}_t + d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t + d\left(\boldsymbol{\alpha}_m\right)\left(-\mathbf{1}_{S\times 1}\sigma c_t^* + \boldsymbol{q}_t^*\right) - \boldsymbol{z}_t\right\},
$$

$$
\begin{aligned}
& L^{-(1+\phi)}\left(\mathcal{W}-\bar{\mathcal{W}}\right)|_{\text{efficient}} \\
=& \frac{1}{2}\sum_{t=0}^{\infty}\beta^t E_0\left[\boldsymbol{c}_t'\left(\left(1-\sigma\right)\boldsymbol{\phi}_l' d\left(\boldsymbol{\alpha}_l\right)^{-1}\boldsymbol{\phi}_c\boldsymbol{\psi}\boldsymbol{\psi}'\right)\boldsymbol{c}_t\right. \\
& -\boldsymbol{c}_t'\left\{d\left(\boldsymbol{\phi}_x\right)d\left(\boldsymbol{\phi}_l\right)d\left(\boldsymbol{\alpha}_l\right)^{-1}+d\left(\boldsymbol{\phi}_c\right)d\left(\boldsymbol{\alpha}_l\right)^{-1}\left(d\left(\boldsymbol{\phi}_l\right)+\phi\boldsymbol{\phi}_l\boldsymbol{\phi}_l'\right)M_l^{-1}\right\}d\left(\boldsymbol{\phi}_c\right)\boldsymbol{c}_t \\
& -\boldsymbol{x}_t'\left\{\left(d\left(\boldsymbol{\theta}^*\right)^{-1}-d\left(\boldsymbol{\phi}_x\right)\right)d\left(\boldsymbol{\phi}_l\right)d\left(\boldsymbol{\alpha}_l\right)^{-1}+d\left(\boldsymbol{\phi}_x\right)d\left(\boldsymbol{\alpha}_l\right)^{-1}\left(d\left(\boldsymbol{\phi}_l\right)+\phi\boldsymbol{\phi}_l\boldsymbol{\phi}_l'\right)M_l^{-1}\right\}d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t \\
& +2\boldsymbol{c}_t'd\left(\boldsymbol{\phi}_c\right)\left\{d\left(\boldsymbol{\phi}_l\right)d\left(\boldsymbol{\alpha}_l\right)^{-1}-d\left(\boldsymbol{\alpha}_l\right)^{-1}\left(d\left(\boldsymbol{\phi}_l\right)+\phi\boldsymbol{\phi}_l\boldsymbol{\phi}_l'\right)M_l^{-1}\right\}d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t \\
& -2\left\{d\left(\boldsymbol{\alpha}_m\right)\left(-\mathbf{1}_{S\times 1}\sigma c_t^*+\boldsymbol{q}_t^*\right)-\boldsymbol{z}_t\right\}'d\left(\boldsymbol{\alpha}_l\right)^{-1}\left(d\left(\boldsymbol{\phi}_l\right)+\phi\boldsymbol{\phi}_l\boldsymbol{\phi}_l'\right)M_l^{-1}d\left(\boldsymbol{\phi}_c\right)\boldsymbol{c}_t \\
& -2\left[\left[\left\{d\left(\boldsymbol{\alpha}_m\right)\left(-\mathbf{1}_{S\times 1}\sigma c_t^*+\boldsymbol{q}_t^*\right)-\boldsymbol{z}_t\right\}'d\left(\boldsymbol{\alpha}_l\right)^{-1}\left(d\left(\boldsymbol{\phi}_l\right)+\phi\boldsymbol{\phi}_l\boldsymbol{\phi}_l'\right)M_l^{-1}\right]\right. \\
& \left.-\left(-\sigma\hat{c}_t^*\mathbf{1}_{S\times 1}+d\left(\boldsymbol{\theta}^*\right)^{-1}\boldsymbol{x}_t^*+\boldsymbol{p}_t^*\right)'d\left(\boldsymbol{\phi}_l\right)d\left(\boldsymbol{\alpha}_l\right)^{-1}\right]d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t \\
& \left.-\boldsymbol{\phi}_l'd\left(\boldsymbol{\alpha}_l\right)^{-1}\left[d\left(\boldsymbol{\theta}\right)d\left(\boldsymbol{\kappa}\right)^{-1}\left\{d\left(\boldsymbol{\phi}_c\right)d\left(\boldsymbol{\pi}_t\right)\boldsymbol{\pi}_t+d\left(\boldsymbol{\phi}_x\right)d\left(\boldsymbol{\pi}_t^X\right)\boldsymbol{\pi}_t^X\right\}\right]+t.i.p.\right.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& L^{-(1+\phi)}\left(\mathcal{W}-\bar{\mathcal{W}}\right) \\
&= E_0\sum_{t=0}^{\infty}\beta^t\left[\frac{1}{2}v_t'\Gamma_{v2}v_t+\xi_t'\Gamma_{\xi v}v_t+\sum_{s\in S}\frac{\theta_s}{2\kappa_s}\left(\Gamma_{\pi s}\pi_{s,t}^2+\Gamma_{\pi s}^X\left(\pi_{s,t}^X\right)^2\right)\right] \\
&+t.i.p.,
\end{aligned}
$$

where

$$
\Gamma_{v2}=\begin{bmatrix}\Gamma_{c2} & \Gamma_{cx}\\ \Gamma_{cx}' & \Gamma_{x2}\end{bmatrix},
$$

$$
\begin{cases}
\Gamma_{c2}= & \left(1-\sigma\right)\boldsymbol{\phi}_l'd\left(\boldsymbol{\alpha}_l\right)^{-1}\boldsymbol{\phi}_c\boldsymbol{\psi}\boldsymbol{\psi}' \\
 & -\left\{d\left(\boldsymbol{\phi}_x\right)d\left(\boldsymbol{\phi}_l\right)d\left(\boldsymbol{\alpha}_l\right)^{-1}+d\left(\boldsymbol{\phi}_c\right)d\left(\boldsymbol{\alpha}_l\right)^{-1}\left(d\left(\boldsymbol{\phi}_l\right)+\phi\boldsymbol{\phi}_l\boldsymbol{\phi}_l'\right)M_l^{-1}\right\}d\left(\boldsymbol{\phi}_c\right) \\
\Gamma_{cx}= & -\phi d\left(\boldsymbol{\phi}_c\right)\boldsymbol{\phi}_l\boldsymbol{\phi}_l'M_l^{-1}d\left(\boldsymbol{\phi}_x\right) \\
\Gamma_{x2}= & -\left\{\left(d\left(\boldsymbol{\theta}^*\right)^{-1}-d\left(\boldsymbol{\phi}_x\right)\right)d\left(\boldsymbol{\phi}_l\right)d\left(\boldsymbol{\alpha}_l\right)^{-1}\right\}d\left(\boldsymbol{\phi}_x\right) \\
 & -\left\{d\left(\boldsymbol{\phi}_x\right)d\left(\boldsymbol{\alpha}_l\right)^{-1}\left(d\left(\boldsymbol{\phi}_l\right)+\phi\boldsymbol{\phi}_l\boldsymbol{\phi}_l'\right)M_l^{-1}\right\}d\left(\boldsymbol{\phi}_x\right)
\end{cases}.
$$

To obtain the expression for $\Gamma_{\xi v}$, note

$$
d\left(\boldsymbol{\alpha}_m\right)\left(-\mathbf{1}_{S\times 1}\sigma c_t^*+\boldsymbol{q}_t^*\right)-\boldsymbol{z}_t=\begin{bmatrix}-\sigma\boldsymbol{\alpha}_m & \mathbb{O}_{S\times S} & \mathbb{O}_{S\times S} & d\left(\boldsymbol{\alpha}_m\right) & -I\end{bmatrix}\xi_t
$$

$$
-\sigma\hat{c}_t^*\mathbf{1}_{S\times 1}+d\left(\boldsymbol{\theta}^*\right)^{-1}\boldsymbol{x}_t^*+\boldsymbol{p}_t^*=\begin{bmatrix}-\sigma\mathbf{1}_{S\times 1} & d\left(\boldsymbol{\theta}^*\right)^{-1} & I & \mathbb{O}_{S\times S} & \mathbb{O}_{S\times S}\end{bmatrix}\xi_t.
$$

Thus,

$$
\Gamma_{\xi v}=\begin{bmatrix}\Gamma_{\xi c} & \Gamma_{\xi x}\end{bmatrix},
$$

$$
\begin{cases}
\Gamma_{\xi c}= & \begin{bmatrix}\sigma\boldsymbol{\alpha}_m & \mathbb{O}_{S\times S} & \mathbb{O}_{S\times S} & -d\left(\boldsymbol{\alpha}_m\right) & I\end{bmatrix}'d\left(\boldsymbol{\alpha}_l\right)^{-1}\left(d\left(\boldsymbol{\phi}_l\right)+\phi\boldsymbol{\phi}_l\boldsymbol{\phi}_l'\right)M_l^{-1}d\left(\boldsymbol{\phi}_c\right) \\
\Gamma_{\xi x}= & \begin{bmatrix}\sigma\boldsymbol{\alpha}_m & \mathbb{O}_{S\times S} & \mathbb{O}_{S\times S} & -d\left(\boldsymbol{\alpha}_m\right) & I\end{bmatrix}'d\left(\boldsymbol{\alpha}_l\right)^{-1}\left(d\left(\boldsymbol{\phi}_l\right)+\phi\boldsymbol{\phi}_l\boldsymbol{\phi}_l'\right)M_l^{-1}d\left(\boldsymbol{\phi}_x\right) \\
 & +\begin{bmatrix}-\sigma\mathbf{1}_{S\times 1} & d\left(\boldsymbol{\theta}^*\right)^{-1} & I & \mathbb{O}_{S\times S} & \mathbb{O}_{S\times S}\end{bmatrix}'d\left(\boldsymbol{\phi}_l\right)d\left(\boldsymbol{\alpha}_l\right)^{-1}d\left(\boldsymbol{\phi}_x\right)
\end{cases}
$$

Now, calculate the flexible price equilibrium to simplify the above expression. The flexible price equilibrium is characterized by

$$\begin{cases} d\left(\boldsymbol{\alpha}_m\right)\left(\mathbf{1}_{S\times 1}q_t^F+\boldsymbol{q}_t^*\right)+\boldsymbol{\alpha}_l w_t^F-\boldsymbol{z}_t-\boldsymbol{p}_t^F=0 \\ d\left(\boldsymbol{\alpha}_m\right)\left(\mathbf{1}_{S\times 1}q_t^F+\boldsymbol{q}_t^*\right)+\boldsymbol{\alpha}_l w_t^F-\boldsymbol{z}_t-\boldsymbol{p}_t^{XF}=0 \end{cases}$$

where

$$\begin{cases} q_t^F=\sigma\left(\boldsymbol{\psi}'\boldsymbol{c}_t^F-c_t^*\right) \\ w_t^F=\phi\boldsymbol{\phi}_l'\boldsymbol{l}_t^F+\sigma\boldsymbol{\psi}'\boldsymbol{c}_t^F \\ \boldsymbol{l}_t^F=M_l^{-1}\left\{d\left(\boldsymbol{\phi}_c\right)\boldsymbol{c}_t^F+d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t^F+d\left(\boldsymbol{\alpha}_m\right)\left(-\mathbf{1}_{S\times 1}\sigma c_t^*+\boldsymbol{q}_t^*\right)-\boldsymbol{z}_t\right\} \\ \boldsymbol{p}_t^F=\mathbf{1}_{S\times 1}\boldsymbol{\psi}'\boldsymbol{c}_t^F-\boldsymbol{c}_t^F \\ \boldsymbol{p}_t^{XF}=-d\left(\boldsymbol{\theta}^*\right)^{-1}\left(\boldsymbol{x}_t^F-\boldsymbol{x}_t^*\right)+\mathbf{1}_{S\times 1}\sigma\left(\boldsymbol{\psi}'\boldsymbol{c}_t^F-c_t^*\right)+\boldsymbol{p}_t^* \end{cases}.$$

First, the pricing equation gives

$$\begin{aligned} \boldsymbol{p}_t^F&=\boldsymbol{p}_t^{XF} \\ \mathbf{1}_{S\times 1}\boldsymbol{\psi}'\boldsymbol{c}_t^F-\boldsymbol{c}_t^F&=-d\left(\boldsymbol{\theta}^*\right)^{-1}\left(\boldsymbol{x}_t^F-\boldsymbol{x}_t^*\right)+\mathbf{1}_{S\times 1}\sigma\left(\boldsymbol{\psi}'\boldsymbol{c}_t^F-\hat{c}_t^*\right)+\boldsymbol{p}_t^* \\ \boldsymbol{x}_t^F&=-d\left(\boldsymbol{\theta}^*\right)\left(\left(1-\sigma\right)\mathbf{1}_{S\times 1}\boldsymbol{\psi}'-I\right)\boldsymbol{c}_t^F+\boldsymbol{x}_t^*-\boldsymbol{\theta}^*\sigma\hat{c}_t^*+d\left(\boldsymbol{\theta}^*\right)\boldsymbol{p}_t^* \end{aligned}$$

Therefore,

$$\begin{aligned} \boldsymbol{l}_t^F&=M_l^{-1}\left\{d\left(\boldsymbol{\phi}_c\right)\boldsymbol{c}_t^F+d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t^F+d\left(\boldsymbol{\alpha}_m\right)\left(-\mathbf{1}_{S\times 1}\sigma c_t^*+\boldsymbol{q}_t^*\right)-\boldsymbol{z}_t\right\} \\ &=M_l^{-1}\left\{d\left(\boldsymbol{\phi}_c\right)-d\left(\boldsymbol{\phi}_x\right)d\left(\boldsymbol{\theta}^*\right)\left(\left(1-\sigma\right)\mathbf{1}_{S\times 1}\boldsymbol{\psi}'-I\right)\right\}\boldsymbol{c}_t^F \\ &+M_l^{-1}\left[-\left(d\left(\boldsymbol{\phi}_x\right)\boldsymbol{\theta}^*+\boldsymbol{\alpha}_m\right)\sigma c_t^*+d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t^*+d\left(\boldsymbol{\phi}_x\right)d\left(\boldsymbol{\theta}^*\right)\boldsymbol{p}_t^*+d\left(\boldsymbol{\alpha}_m\right)\boldsymbol{q}_t^*-\boldsymbol{z}_t\right] \end{aligned}$$

Thus,

$$d\left(\boldsymbol{\alpha}_m\right)\left(\mathbf{1}_{S\times 1}q_t^F+\hat{\boldsymbol{q}}_t^*\right)+\boldsymbol{\alpha}_l w_t^F-\hat{\boldsymbol{z}}_t=\boldsymbol{p}_t^F$$

$$\begin{aligned} &d\left(\boldsymbol{\alpha}_m\right)\left(\mathbf{1}_{S\times 1}\sigma\left(\boldsymbol{\psi}'\boldsymbol{c}_t^F-\hat{c}_t^*\right)+\hat{\boldsymbol{q}}_t^*\right)+\boldsymbol{\alpha}_l\left(\phi\boldsymbol{\phi}_l'\boldsymbol{l}_t^F+\sigma\boldsymbol{\psi}'\boldsymbol{c}_t^F\right)-\hat{\boldsymbol{z}}_t \\ &=\mathbf{1}_{S\times 1}\boldsymbol{\psi}'\boldsymbol{c}_t^F-\boldsymbol{c}_t^F \end{aligned}$$

$$\begin{aligned} &d\left(\boldsymbol{\alpha}_m\right)\mathbf{1}_{S\times 1}\sigma\boldsymbol{\psi}'\boldsymbol{c}_t^F+\boldsymbol{\alpha}_l\sigma\boldsymbol{\psi}'\boldsymbol{c}_t^F-\left(\mathbf{1}_{S\times 1}\boldsymbol{\psi}'-I\right)\boldsymbol{c}_t^F \\ &=\boldsymbol{\alpha}_m\sigma\hat{c}_t^*-d\left(\boldsymbol{\alpha}_m\right)\hat{\boldsymbol{q}}_t^*-\boldsymbol{\alpha}_l\phi\boldsymbol{\phi}_l'\boldsymbol{l}_t^F+\hat{\boldsymbol{z}}_t \end{aligned}$$

$$\begin{aligned} &d\left(\boldsymbol{\alpha}_m\right)\mathbf{1}_{S\times 1}\sigma\boldsymbol{\psi}'\boldsymbol{c}_t^F+\boldsymbol{\alpha}_l\sigma\boldsymbol{\psi}'\boldsymbol{c}_t^F-\left(\mathbf{1}_{S\times 1}\boldsymbol{\psi}'-I\right)\boldsymbol{c}_t^F \\ &+\boldsymbol{\alpha}_l\phi\boldsymbol{\phi}_l'M_l^{-1}\left\{d\left(\boldsymbol{\phi}_c\right)-d\left(\boldsymbol{\phi}_x\right)d\left(\boldsymbol{\theta}^*\right)\left(\left(1-\sigma\right)\mathbf{1}_{S\times 1}\boldsymbol{\psi}'-I\right)\right\}\boldsymbol{c}_t^F \\ =&\boldsymbol{\alpha}_m\sigma\hat{c}_t^*-d\left(\boldsymbol{\alpha}_m\right)\hat{\boldsymbol{q}}_t^*+\hat{\boldsymbol{z}}_t-\boldsymbol{\alpha}_l\phi\boldsymbol{\phi}_l'M_l^{-1} \\ &\times\left[-\left(d\left(\boldsymbol{\phi}_x\right)\boldsymbol{\theta}^*+\boldsymbol{\alpha}_m\right)\sigma c_t^*+d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t^*+d\left(\boldsymbol{\phi}_x\right)d\left(\boldsymbol{\theta}^*\right)\boldsymbol{p}_t^*+d\left(\boldsymbol{\alpha}_m\right)\boldsymbol{q}_t^*-\boldsymbol{z}_t\right] \end{aligned}$$

$$
\begin{aligned}
&\left[(\sigma-1)\,\mathbf{1}_{S\times 1}\boldsymbol{\psi}'+I\right]\boldsymbol{c}_t^F\\
&+\left[\boldsymbol{\alpha}_l\phi\boldsymbol{\phi}_l'M_l^{-1}\left\{d\left(\boldsymbol{\phi}_c\right)-d\left(\boldsymbol{\phi}_x\right)d\left(\boldsymbol{\theta}^*\right)\left((1-\sigma)\,\mathbf{1}_{S\times 1}\boldsymbol{\psi}'-I\right)\right\}\right]\boldsymbol{c}_t^F\\
=&\boldsymbol{\alpha}_m\sigma\hat{c}_t^*-d\left(\boldsymbol{\alpha}_m\right)\hat{\boldsymbol{q}}_t^*+\hat{\boldsymbol{z}}_t-\boldsymbol{\alpha}_l\phi\boldsymbol{\phi}_l'M_l^{-1}\\
&\times\left[-\left(d\left(\boldsymbol{\phi}_x\right)\boldsymbol{\theta}^*+\boldsymbol{\alpha}_m\right)\sigma c_t^*+d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t^*+d\left(\boldsymbol{\phi}_x\right)d\left(\boldsymbol{\theta}^*\right)\boldsymbol{p}_t^*+d\left(\boldsymbol{\alpha}_m\right)\boldsymbol{q}_t^*-\boldsymbol{z}_t\right]
\end{aligned}
$$

That is,

$$
\begin{aligned}
&\left[(\sigma-1)\,\mathbf{1}_{S\times 1}\boldsymbol{\psi}'+I+\boldsymbol{\alpha}_l\phi\boldsymbol{\phi}_l'M_l^{-1}\left\{d\left(\boldsymbol{\phi}_c\right)-d\left(\boldsymbol{\phi}_x\right)d\left(\boldsymbol{\theta}^*\right)\left((1-\sigma)\,\mathbf{1}_{S\times 1}\boldsymbol{\psi}'-I\right)\right\}\right]\boldsymbol{c}_t^F\\
&=\left[\boldsymbol{\alpha}_m+\boldsymbol{\alpha}_l\phi\boldsymbol{\phi}_l'M_l^{-1}\left(d\left(\boldsymbol{\phi}_x\right)\boldsymbol{\theta}^*+\boldsymbol{\alpha}_m\right)\right]\sigma c_t^*-\boldsymbol{\alpha}_l\phi\boldsymbol{\phi}_l'M_l^{-1}d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t^*\\
&-\boldsymbol{\alpha}_l\phi\boldsymbol{\phi}_l'M_l^{-1}d\left(\boldsymbol{\phi}_x\right)d\left(\boldsymbol{\theta}^*\right)\boldsymbol{p}_t^*-\left[I+\boldsymbol{\alpha}_l\phi\boldsymbol{\phi}_l'M_l^{-1}\right]d\left(\boldsymbol{\alpha}_m\right)\boldsymbol{q}_t^*+\left[I+\boldsymbol{\alpha}_l\phi\boldsymbol{\phi}_l'M_l^{-1}\right]\boldsymbol{z}_t
\end{aligned}
$$

$$
\boldsymbol{c}_t^F=M_{cc}^{-1}M_{c\xi}\xi_t
$$

$$
M_{cc}=\left[(\sigma-1)\,\mathbf{1}_{S\times 1}\boldsymbol{\psi}'+I+\boldsymbol{\alpha}_l\phi\boldsymbol{\phi}_l'M_l^{-1}\left\{d\left(\boldsymbol{\phi}_c\right)-d\left(\boldsymbol{\phi}_x\right)d\left(\boldsymbol{\theta}^*\right)\left((1-\sigma)\,\mathbf{1}_{S\times 1}\boldsymbol{\psi}'-I\right)\right\}\right]
$$

$$
M_{c\xi}=\left[\begin{array}{ccccc} M_{cc*} & M_{cx*} & M_{cp*} & M_{cq*} & M_{cz}\end{array}\right].
$$

$$
\begin{cases}
M_{cc*}=\left[\boldsymbol{\alpha}_m+\boldsymbol{\alpha}_l\phi\boldsymbol{\phi}_l'M_l^{-1}\left(d\left(\boldsymbol{\phi}_x\right)\boldsymbol{\theta}^*+\boldsymbol{\alpha}_m\right)\right]\sigma\\
M_{cx*}=-\boldsymbol{\alpha}_l\phi\boldsymbol{\phi}_l'M_l^{-1}d\left(\boldsymbol{\phi}_x\right)\\
M_{cp*}=-\boldsymbol{\alpha}_l\phi\boldsymbol{\phi}_l'M_l^{-1}d\left(\boldsymbol{\phi}_x\right)d\left(\boldsymbol{\theta}^*\right)\\
M_{cq*}=-\left[I+\boldsymbol{\alpha}_l\phi\boldsymbol{\phi}_l'M_l^{-1}\right]d\left(\boldsymbol{\alpha}_m\right)\\
M_{cz}=\left[I+\boldsymbol{\alpha}_l\phi\boldsymbol{\phi}_l'M_l^{-1}\right]
\end{cases}
$$

$$
\begin{aligned}
\boldsymbol{x}_t^F&=-d\left(\boldsymbol{\theta}^*\right)\left((1-\sigma)\,\mathbf{1}_{S\times 1}\boldsymbol{\psi}'-I\right)M_{cc}^{-1}M_{c\xi}\xi_t+\left[\begin{array}{ccccc}-\boldsymbol{\theta}^*\sigma & I & d\left(\boldsymbol{\theta}^*\right) & \mathbb{O}_{S\times S} & \mathbb{O}_{S\times S}\end{array}\right]\xi_t\\
&=\left\{-d\left(\boldsymbol{\theta}^*\right)\left((1-\sigma)\,\mathbf{1}_{S\times 1}\boldsymbol{\psi}'-I\right)M_{cc}^{-1}M_{c\xi}+\left[\begin{array}{ccccc}-\boldsymbol{\theta}^*\sigma & I & d\left(\boldsymbol{\theta}^*\right) & \mathbb{O}_{S\times S} & \mathbb{O}_{S\times S}\end{array}\right]\right\}\xi_t
\end{aligned}
$$

In terms of $v_t^F$,

$$
v_t^F=\left[\begin{array}{c}\boldsymbol{c}_t^F\\ \boldsymbol{x}_t^F\end{array}\right]=\underbrace{\left[\begin{array}{c}M_{cc}^{-1}M_{c\xi}\\ \left\{-d\left(\boldsymbol{\theta}^*\right)\left((1-\sigma)\,\mathbf{1}_{S\times 1}\boldsymbol{\psi}'-I\right)M_{cc}^{-1}M_{c\xi}+\left[\begin{array}{ccccc}-\boldsymbol{\theta}^*\sigma & I & d\left(\boldsymbol{\theta}^*\right) & \mathbb{O}_{S\times S} & \mathbb{O}_{S\times S}\end{array}\right]\right\}\end{array}\right]}_{=:F}\xi_t
$$

Defining

$$
F_c=M_{cc}^{-1}M_{c\xi},\;F_x=\left\{-d\left(\boldsymbol{\theta}^*\right)\left((1-\sigma)\,\mathbf{1}_{S\times 1}\boldsymbol{\psi}'-I\right)M_{cc}^{-1}M_{c\xi}+\left[\begin{array}{ccccc}-\boldsymbol{\theta}^*\sigma & I & d\left(\boldsymbol{\theta}^*\right) & \mathbb{O}_{S\times S} & \mathbb{O}_{S\times S}\end{array}\right]\right\},
$$

$$
F=\left[F_c',F_x'\right]'.
$$

The following shows that the second-order approximated welfare can be expressed in the quadratic form of the gap from the flexible price equilibrium. That is,

$$
\begin{cases}
\Gamma_{c2}F_c+\Gamma_{cx}F_x+\Gamma_{\xi c}'=0 & (a)'\\
\Gamma_{cx}'F_c+\Gamma_{x2}F_x+\Gamma_{\xi x}'=0 & (b)'
\end{cases}.
$$

This set of equations is sufficient to see that when we express the real terms $\frac{1}{2}v_t'\Gamma_{v2}v_t+\xi_t'\Gamma_{\xi v}v_t$ in the deviations from the natural level as

$$\frac{1}{2}\tilde{v}_t'\Gamma_{v2}\tilde{v}_t,$$

where $\tilde{v}_t := v_t - v_t^{Nat}$, the natural level $v_t^{Nat}$ coincides with the flexible price equilibrium since

$$\frac{1}{2}\tilde{v}_t'\Gamma_{v2}\tilde{v}_t = \frac{1}{2}\left(v_t - N\xi_t\right)'\Gamma_{v2}\left(v_t - N\xi_t\right) = \frac{1}{2}v_t'\Gamma_{v2}v_t - \xi_t'N'\Gamma_{v2}v_t + t.i.p.$$

Part (a)'

$$\begin{aligned}&\Gamma_{c2}F_c + \Gamma_{cx}F_x \\ =&-d\left(\boldsymbol{\phi}_c\right)d\left(\boldsymbol{\phi}_l\right)d\left(\boldsymbol{\alpha}_l\right)^{-1}\begin{bmatrix} M_{cc*} & M_{cx*} & M_{cp*} & M_{cq*} & M_{cz}\end{bmatrix} \\ &-\phi d\left(\boldsymbol{\phi}_c\right)\boldsymbol{\phi}_l\boldsymbol{\phi}_l'M_l^{-1}d\left(\boldsymbol{\phi}_x\right)\begin{bmatrix} -\boldsymbol{\theta}^*\sigma & I & d\left(\boldsymbol{\theta}^*\right) & \mathbb{O}_{S\times S} & \mathbb{O}_{S\times S}\end{bmatrix}\end{aligned}$$

The second to $S+1$th columns of part (a)' is

$$\begin{aligned}&-d\left(\boldsymbol{\phi}_c\right)d\left(\boldsymbol{\phi}_l\right)d\left(\boldsymbol{\alpha}_l\right)^{-1}M_{cx*} - \phi d\left(\boldsymbol{\phi}_c\right)\boldsymbol{\phi}_l\boldsymbol{\phi}_l'M_l^{-1}d\left(\boldsymbol{\phi}_x\right) \\ =&d\left(\boldsymbol{\phi}_c\right)d\left(\boldsymbol{\phi}_l\right)d\left(\boldsymbol{\alpha}_l\right)^{-1}\boldsymbol{\alpha}_l\phi\boldsymbol{\phi}_l'M_l^{-1}d\left(\boldsymbol{\phi}_x\right) - \phi d\left(\boldsymbol{\phi}_c\right)\boldsymbol{\phi}_l\boldsymbol{\phi}_l'M_l^{-1}d\left(\boldsymbol{\phi}_x\right) \\ =&d\left(\boldsymbol{\phi}_c\right)d\left(\boldsymbol{\phi}_l\right)d\left(\boldsymbol{\alpha}_l\right)^{-1}\left(\boldsymbol{\alpha}_l\phi - \phi\boldsymbol{\alpha}_l\right)\boldsymbol{\phi}_l'M_l^{-1}d\left(\boldsymbol{\phi}_x\right) \\ =&0\end{aligned}$$

The $S+2$ to $2S+1$ columns of part (a)' are

$$\begin{aligned}&-d\left(\boldsymbol{\phi}_c\right)d\left(\boldsymbol{\phi}_l\right)d\left(\boldsymbol{\alpha}_l\right)^{-1}M_{cp*} - \phi d\left(\boldsymbol{\phi}_c\right)\boldsymbol{\phi}_l\boldsymbol{\phi}_l'M_l^{-1}d\left(\boldsymbol{\phi}_x\right)d\left(\boldsymbol{\theta}^*\right) \\ =&d\left(\boldsymbol{\phi}_c\right)d\left(\boldsymbol{\phi}_l\right)d\left(\boldsymbol{\alpha}_l\right)^{-1}\left(\boldsymbol{\alpha}_l\phi - \phi\boldsymbol{\alpha}_l\right)\boldsymbol{\phi}_l'M_l^{-1}d\left(\boldsymbol{\phi}_x\right)d\left(\boldsymbol{\theta}^*\right) \\ =&0\end{aligned}$$

The $2S+2$ to $3S+1$ columns of part (a)' are

$$\begin{aligned}&-d\left(\boldsymbol{\phi}_c\right)d\left(\boldsymbol{\phi}_l\right)d\left(\boldsymbol{\alpha}_l\right)^{-1}M_{cq*} \\ &-d\left(\boldsymbol{\phi}_c\right)\left(M_l^{-1}\right)'\left(d\left(\boldsymbol{\phi}_l\right)+\phi\boldsymbol{\phi}_l\boldsymbol{\phi}_l'\right)'d\left(\boldsymbol{\alpha}_l\right)^{-1}d\left(\boldsymbol{\alpha}_m\right) \\ &= d\left(\boldsymbol{\phi}_c\right)\left\{\underbrace{d\left(\boldsymbol{\phi}_l\right)d\left(\boldsymbol{\alpha}_l\right)^{-1}\left(I+\phi\mathbf{1}_{S\times1}\boldsymbol{\phi}_l'\right)M_l^{-1}}_{(*)}\right\}d\left(\boldsymbol{\alpha}_m\right) \\ &-d\left(\boldsymbol{\phi}_c\right)\left\{\underbrace{\left\{d\left(\boldsymbol{\alpha}_l\right)^{-1}d\left(\boldsymbol{\phi}_l\right)\left(I+\phi\mathbf{1}_{S\times1}\boldsymbol{\phi}_l'\right)M_l^{-1}\right\}'}_{=(*)'}\right\}d\left(\boldsymbol{\alpha}_m\right)\end{aligned}$$

It suffices to show that $d\left(\boldsymbol{\phi}_l\right)d\left(\boldsymbol{\alpha}_l\right)^{-1}\left(I+\phi\mathbf{1}_{S\times1}\boldsymbol{\phi}_l'\right)M_l^{-1}$ is symmetric, but it is indeed symmetric.

The last $S$ columns of part (a)' are

$$
\begin{aligned}
&-d(\boldsymbol{\phi}_c)\,d(\boldsymbol{\phi}_l)\,d(\boldsymbol{\alpha}_l)^{-1} M_{cz} + \left(d(\boldsymbol{\alpha}_l)^{-1}\left(d(\boldsymbol{\phi}_l) + \phi\boldsymbol{\phi}_l\boldsymbol{\phi}_l'\right) M_l^{-1} d(\boldsymbol{\phi}_c)\right)' \\
=&-d(\boldsymbol{\phi}_c)\,d(\boldsymbol{\phi}_l)\,d(\boldsymbol{\alpha}_l)^{-1}\left(I + \boldsymbol{\alpha}_l\phi\boldsymbol{\phi}_l' M_l^{-1}\right) + d(\boldsymbol{\phi}_c)\left(\left(I + \phi\mathbf{1}_{S\times 1}\boldsymbol{\phi}_l'\right) M_l^{-1}\right)' d(\boldsymbol{\phi}_l)\,d(\boldsymbol{\alpha}_l)^{-1} \\
=&-d(\boldsymbol{\phi}_c)\left\{\underbrace{d(\boldsymbol{\phi}_l)\,d(\boldsymbol{\alpha}_l)^{-1}\left(I + \phi\mathbf{1}_{S\times 1}\boldsymbol{\phi}_l'\right) M_l^{-1}}_{=(*)} - \underbrace{\left\{d(\boldsymbol{\phi}_l)\,d(\boldsymbol{\alpha}_l)^{-1}\left(\left(I + \phi\mathbf{1}_{S\times 1}\boldsymbol{\phi}_l'\right) M_l^{-1}\right)\right\}'}_{=(*)'}\right\}
\end{aligned}
$$

Part (b)'
Therefore,

$$
\begin{aligned}
&\Gamma_{cx}' F_c + \Gamma_{x2} F_x \\
=&-d(\boldsymbol{\phi}_l)\,d(\boldsymbol{\alpha}_l)^{-1} d(\boldsymbol{\phi}_x)\, M_{c\xi} \\
&-\left\{d(\boldsymbol{\theta}^*)^{-1} d(\boldsymbol{\phi}_l)\,d(\boldsymbol{\alpha}_l)^{-1} + d(\boldsymbol{\phi}_x)\,d(\boldsymbol{\phi}_l)\,\phi\mathbf{1}_{S\times 1}\boldsymbol{\phi}_l' M_l^{-1}\right\} \\
&\times d(\boldsymbol{\phi}_x)\begin{bmatrix} -\boldsymbol{\theta}^*\sigma & I & d(\boldsymbol{\theta}^*) & \mathbb{O}_{S\times S} & \mathbb{O}_{S\times S} \end{bmatrix}
\end{aligned}
$$

The first column of part (b)' is

$$
\begin{aligned}
&-d(\boldsymbol{\phi}_l)\,d(\boldsymbol{\alpha}_l)^{-1} d(\boldsymbol{\phi}_x)\left[\boldsymbol{\alpha}_m + \boldsymbol{\alpha}_l\phi\boldsymbol{\phi}_l' M_l^{-1}\left(d(\boldsymbol{\phi}_x)\,\boldsymbol{\theta}^* + \boldsymbol{\alpha}_m\right)\right]\sigma \\
&+\left\{d(\boldsymbol{\theta}^*)^{-1} d(\boldsymbol{\phi}_l)\,d(\boldsymbol{\alpha}_l)^{-1} + d(\boldsymbol{\phi}_x)\,d(\boldsymbol{\phi}_l)\,\phi\mathbf{1}_{S\times 1}\boldsymbol{\phi}_l' M_l^{-1}\right\} d(\boldsymbol{\phi}_x)\,\boldsymbol{\theta}^*\sigma \\
&+d(\boldsymbol{\phi}_x)\left(M_l^{-1}\right)'\left(d(\boldsymbol{\phi}_l) + \phi\boldsymbol{\phi}_l\boldsymbol{\phi}_l'\right) d(\boldsymbol{\alpha}_l)^{-1}\sigma\boldsymbol{\alpha}_m - d(\boldsymbol{\phi}_x)\,d(\boldsymbol{\alpha}_l)^{-1} d(\boldsymbol{\phi}_l)\,\mathbf{1}_{S\times 1}\sigma \\
=&-d(\boldsymbol{\phi}_x)\left\{\underbrace{d(\boldsymbol{\phi}_l)\,d(\boldsymbol{\alpha}_l)^{-1}\left(I + \phi\mathbf{1}_{S\times 1}\boldsymbol{\phi}_l'\right) M_l^{-1}}_{=(*)}\right\}\sigma\boldsymbol{\alpha}_m \\
&+d(\boldsymbol{\phi}_x)\left\{\underbrace{\left(d(\boldsymbol{\alpha}_l)^{-1} d(\boldsymbol{\phi}_l)\left(I + \phi\mathbf{1}_{S\times 1}\boldsymbol{\phi}_l'\right) M_l^{-1}\right)'}_{=(*)'}\right\}\sigma\boldsymbol{\alpha}_m \\
&-d(\boldsymbol{\phi}_x)\,d(\boldsymbol{\phi}_l)\left\{\mathbf{1}_{S\times 1}\phi - \phi\mathbf{1}_{S\times 1}\right\}\boldsymbol{\phi}_l' M_l^{-1} d(\boldsymbol{\phi}_x)\,\boldsymbol{\theta}^*\sigma
\end{aligned}
$$

The 2nd to $S+1$'th columns of part (b)' are

$$
\begin{aligned}
&d(\boldsymbol{\phi}_l)\,d(\boldsymbol{\alpha}_l)^{-1} d(\boldsymbol{\phi}_x)\,\boldsymbol{\alpha}_l\phi\boldsymbol{\phi}_l' M_l^{-1} d(\boldsymbol{\phi}_x) \\
&-\left\{d(\boldsymbol{\theta}^*)^{-1} d(\boldsymbol{\phi}_l)\,d(\boldsymbol{\alpha}_l)^{-1} + d(\boldsymbol{\phi}_x)\,d(\boldsymbol{\phi}_l)\,\phi\mathbf{1}_{S\times 1}\boldsymbol{\phi}_l' M_l^{-1}\right\} d(\boldsymbol{\phi}_x) \\
&+d(\boldsymbol{\phi}_x)\,d(\boldsymbol{\alpha}_l)^{-1} d(\boldsymbol{\phi}_l)\,d(\boldsymbol{\theta}^*)^{-1} \\
=&d(\boldsymbol{\phi}_x)\,d(\boldsymbol{\phi}_l)\,\mathbf{1}_{S\times 1}\phi\boldsymbol{\phi}_l' M_l^{-1} d(\boldsymbol{\phi}_x) \\
&-d(\boldsymbol{\phi}_x)\,d(\boldsymbol{\phi}_l)\,\phi\mathbf{1}_{S\times 1}\boldsymbol{\phi}_l' M_l^{-1} d(\boldsymbol{\phi}_x) \\
=&0
\end{aligned}
$$

The $S+2$th to $2S+1$th columns of part (b)' are

$$\begin{aligned}
&d\left(\boldsymbol{\phi}_l\right) d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(\boldsymbol{\phi}_x\right) \boldsymbol{\alpha}_l \phi \boldsymbol{\phi}_l' M_l^{-1} d\left(\boldsymbol{\phi}_x\right) d\left(\boldsymbol{\theta}^*\right) \\
&- \left\{d\left(\boldsymbol{\theta}^*\right)^{-1} d\left(\boldsymbol{\phi}_l\right) d\left(\boldsymbol{\alpha}_l\right)^{-1} + d\left(\boldsymbol{\phi}_x\right) d\left(\boldsymbol{\phi}_l\right) \phi \mathbf{1}_{S\times 1} \boldsymbol{\phi}_l' M_l^{-1}\right\} d\left(\boldsymbol{\phi}_x\right) d\left(\boldsymbol{\theta}^*\right) \\
&+ d\left(\boldsymbol{\phi}_x\right) d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(\boldsymbol{\phi}_l\right) \\
=& d\left(\boldsymbol{\phi}_l\right) d\left(\boldsymbol{\phi}_x\right) \phi \mathbf{1}_{S\times 1} \boldsymbol{\phi}_l' M_l^{-1} d\left(\boldsymbol{\phi}_x\right) d\left(\boldsymbol{\theta}^*\right) \\
&- d\left(\boldsymbol{\phi}_x\right) d\left(\boldsymbol{\phi}_l\right) \phi \mathbf{1}_{S\times 1} \boldsymbol{\phi}_l' M_l^{-1} d\left(\boldsymbol{\phi}_x\right) d\left(\boldsymbol{\theta}^*\right) \\
=& 0
\end{aligned}$$

The $2S+2$'th to $3S+1$'th columns of part (b)' are

$$\begin{aligned}
&d\left(\boldsymbol{\phi}_l\right) d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(\boldsymbol{\phi}_x\right) \left[I + \boldsymbol{\alpha}_l \phi \boldsymbol{\phi}_l' M_l^{-1}\right] d\left(\boldsymbol{\alpha}_m\right) \\
&- d\left(\boldsymbol{\phi}_x\right) \left(M_l^{-1}\right)' \left(d\left(\boldsymbol{\phi}_l\right) + \phi \boldsymbol{\phi}_l \boldsymbol{\phi}_l'\right) d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(\boldsymbol{\alpha}_m\right) \\
=& d\left(\boldsymbol{\phi}_x\right) \left\{ \underbrace{d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(\boldsymbol{\phi}_l\right) \left(I + \phi \mathbf{1}_{S\times 1} \boldsymbol{\phi}_l'\right) M_l^{-1}}_{=(*)} \right\} d\left(\boldsymbol{\alpha}_m\right) \\
&- d\left(\boldsymbol{\phi}_x\right) \left\{ \underbrace{\left(d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(\boldsymbol{\phi}_l\right) \left(I + \phi \mathbf{1}_{S\times 1} \boldsymbol{\phi}_l'\right) M_l^{-1}\right)'}_{=(*)'} \right\} d\left(\boldsymbol{\alpha}_m\right)
\end{aligned}$$

The last $S$ columns of part (b)' are

$$\begin{aligned}
&- d\left(\boldsymbol{\phi}_l\right) d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(\boldsymbol{\phi}_x\right) \left(I + \boldsymbol{\alpha}_l \phi \boldsymbol{\phi}_l' M_l^{-1}\right) \\
&+ d\left(\boldsymbol{\phi}_x\right) \left(M_l^{-1}\right)' \left(d\left(\boldsymbol{\phi}_l\right) + \phi \boldsymbol{\phi}_l \boldsymbol{\phi}_l'\right) d\left(\boldsymbol{\alpha}_l\right)^{-1} \\
=& - d\left(\boldsymbol{\phi}_x\right) \left\{ \underbrace{d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(\boldsymbol{\phi}_l\right) \left(I + \phi \mathbf{1}_{S\times 1} \boldsymbol{\phi}_l'\right) M_l^{-1}}_{=(*)} \right\} \\
&+ d\left(\boldsymbol{\phi}_x\right) \left\{ \underbrace{\left(d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(\boldsymbol{\phi}_l\right) \left(I + \phi \mathbf{1}_{S\times 1} \boldsymbol{\phi}_l'\right) M_l^{-1}\right)'}_{=(*)'} \right\}.
\end{aligned}$$

## B.7 Proof of Lemma 4

From Appendix B.6, we can see that, under the efficient steady state, the objective function is approximated purely quadratically by

$$\begin{aligned}
&\mathcal{W} - \bar{\mathcal{W}} \\
&\propto E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{1}{2} \tilde{v}_t' \Gamma_{v2} \tilde{v}_t + \sum_{s\in S} \frac{\theta_s}{2\kappa_s} \left(\Gamma_{\pi s} \pi_{s,t}^2 + \Gamma_{\pi s}^X \left(\pi_{s,t}^X\right)^2\right)\right] \\
&+ t.i.p.,
\end{aligned}$$

where $\tilde{v}_t = v_t - v_t^F = v_t - \left[\left(\boldsymbol{c}_t^F\right)', \left(\boldsymbol{x}_t^F\right)'\right]'$. It remains to show the form of the constraints.

All the constraints in Definition 1 except for pricing equations (13)-(18), (21), and (22) are already used to substitute out auxiliary endogenous variables. The linear approximations of these pricing equations reduce to the Phillips curve for each sector.

$$\begin{aligned}
&\kappa_s^{-1}\left(\pi_{s,t} - \beta E_t\left[\pi_{s,t+1}\right]\right) = \alpha_{sm}\left(q_t + q_{st}^*\right) + \alpha_{sl} w_t - z_{st} - p_{st} \\
&\kappa_s^{-1}\left(\pi_{s,t}^X - \beta E_t\left[\pi_{s,t+1}^X\right]\right) = \alpha_{sm}\left(q_t + q_{st}^*\right) + \alpha_{sl} w_t - z_{st} - p_{st}^X \\
&\pi_{s,t} = \pi_t + p_{st} - p_{st-1} \\
&\pi_{s,t}^X = \pi_t + p_{st}^X - p_{st-1}^X
\end{aligned}$$

In matrix,

$$\begin{aligned}
&d\left(\boldsymbol{\kappa}\right)^{-1}\left(\boldsymbol{\pi}_t - \beta E_t\left[\boldsymbol{\pi}_{t+1}\right]\right) = d\left(\boldsymbol{\alpha}_m\right)\left(\mathbf{1}_{S\times 1} q_t + \boldsymbol{q}_t^*\right) + \boldsymbol{\alpha}_l w_t - \boldsymbol{z}_t - \boldsymbol{p}_t \\
&d\left(\boldsymbol{\kappa}\right)^{-1}\left(\boldsymbol{\pi}_t^X - \beta E_t\left[\boldsymbol{\pi}_{t+1}^X\right]\right) = d\left(\boldsymbol{\alpha}_m\right)\left(\mathbf{1}_{S\times 1} q_t + \boldsymbol{q}_t^*\right) + \boldsymbol{\alpha}_l w_t - \boldsymbol{z}_t - \boldsymbol{p}_t^X \\
&\boldsymbol{\pi}_t = \mathbf{1}_{S\times 1}\pi_t + \boldsymbol{p}_t - \boldsymbol{p}_{t-1} \\
&\boldsymbol{\pi}_t^X = \mathbf{1}_{S\times 1}\pi_t + \boldsymbol{p}_t^X - \boldsymbol{p}_{t-1}^X.
\end{aligned}$$

Comparing this with the condition of the flexible price equilibrium:

$$\begin{aligned}
d\left(\boldsymbol{\alpha}_m\right)\left(\mathbf{1}_{S\times 1} q_t^F + \boldsymbol{q}_t^*\right) + \boldsymbol{\alpha}_l w_t^F - \boldsymbol{z}_t - \boldsymbol{p}_t^F = 0 \\
d\left(\boldsymbol{\alpha}_m\right)\left(\mathbf{1}_{S\times 1} q_t^F + \boldsymbol{q}_t^*\right) + \boldsymbol{\alpha}_l w_t^F - \boldsymbol{z}_t - \boldsymbol{p}_t^{XF} = 0,
\end{aligned}$$

the Phillips curves can be rewritten as

$$d\left(\boldsymbol{\kappa}\right)^{-1}\left(\boldsymbol{\pi}_t - \beta E_t\left[\boldsymbol{\pi}_{t+1}\right]\right) = d\left(\boldsymbol{\alpha}_m\right)\mathbf{1}_{S\times 1}\tilde{q}_t + \boldsymbol{\alpha}_l \tilde{w}_t - \tilde{\boldsymbol{p}}_t \tag{33}$$

$$d\left(\boldsymbol{\kappa}\right)^{-1}\left(\boldsymbol{\pi}_t^X - \beta E_t\left[\boldsymbol{\pi}_{t+1}^X\right]\right) = d\left(\boldsymbol{\alpha}_m\right)\mathbf{1}_{S\times 1}\tilde{q}_t + \boldsymbol{\alpha}_l \tilde{w}_t - \tilde{\boldsymbol{p}}_t^X. \tag{34}$$

Since the linear approximation of other equilibrium conditions that map $q_t, w_t, \boldsymbol{p}_t, \boldsymbol{p}_t^X$ into $\boldsymbol{c}_t, \boldsymbol{x}_t$ hold both in the sticky price equilibrium and in the flexible price equilibrium, the gap on the right hand side is linear in $\tilde{\boldsymbol{c}}_t$ and $\tilde{\boldsymbol{x}}_t$

$$\begin{cases}
\tilde{q}_t = \sigma\boldsymbol{\psi}'\tilde{\boldsymbol{c}}_t \\
\tilde{w}_t = \phi\boldsymbol{\phi}_l'\tilde{\boldsymbol{l}}_t + \sigma\boldsymbol{\psi}'\tilde{\boldsymbol{c}}_t \\
\tilde{\boldsymbol{l}}_t = M_l^{-1}\left\{d\left(\boldsymbol{\phi}_c\right)\tilde{\boldsymbol{c}}_t + d\left(\boldsymbol{\phi}_x\right)\tilde{\boldsymbol{x}}_t\right\} \\
\tilde{\boldsymbol{p}}_t = \mathbf{1}_{S\times 1}\boldsymbol{\psi}'\tilde{\boldsymbol{c}}_t - \tilde{\boldsymbol{c}}_t \\
\tilde{\boldsymbol{p}}_t^X = -d\left(\boldsymbol{\theta}^*\right)^{-1}\tilde{\boldsymbol{x}}_t + \mathbf{1}_{S\times 1}\sigma\boldsymbol{\psi}'\tilde{\boldsymbol{c}}_t
\end{cases}.$$

Plugging these into the Phillips curve (33) and (34), we can find $\gamma_v^P$ and $\gamma_{Xv}^P$ in the following expressions.

$$\begin{aligned}
d\left(\boldsymbol{\kappa}\right)^{-1}\left(\boldsymbol{\pi}_t - \beta E_t\left[\boldsymbol{\pi}_{t+1}\right]\right) = \gamma_v^P\tilde{v}_t \\
d\left(\boldsymbol{\kappa}\right)^{-1}\left(\boldsymbol{\pi}_t^X - \beta E_t\left[\boldsymbol{\pi}_{t+1}^X\right]\right) = \gamma_{Xv}^P\tilde{v}_t.
\end{aligned}$$

For the identity, we can rewrite

$$\boldsymbol{\pi}_t = \mathbf{1}_{S\times 1}\pi_t + \tilde{\boldsymbol{p}}_t - \tilde{\boldsymbol{p}}_{t-1} + \boldsymbol{p}_t^F - \boldsymbol{p}_{t-1}^F$$
$$\boldsymbol{\pi}_t^X = \mathbf{1}_{S\times 1}\pi_t + \tilde{\boldsymbol{p}}_t^X - \tilde{\boldsymbol{p}}_{t-1}^X + \boldsymbol{p}_t^{XF} - \boldsymbol{p}_{t-1}^{XF}.$$

The gaps $\tilde{\boldsymbol{p}}_t$ and $\tilde{\boldsymbol{p}}_t^X$ can be similarly rewritten in terms of $\tilde{\boldsymbol{c}}_t$ and $\tilde{\boldsymbol{x}}_t$. This gives the expressions for $\gamma_v^I$ and $\gamma_{vX}^I$. Regarding the flexible price equilibrium objects, $\boldsymbol{p}_t^F$ and $\boldsymbol{p}_t^{XF}$, substitute the solutions as functions of exogenous variables. This gives the expressions for $\epsilon_t^I$ and $\epsilon_t^{IX}$.

## B.8 Solution in the long-run expectation

This section derives the RPI as the index whose long-run expectation remains constant under the optimal monetary policy. The argument parallels that in Woodford [2010]. To this end, I take the first-order condition of the approximated Ramsey problem given in Lemma 4.

$$\begin{cases} [\tilde{v}_t] & \Gamma_{v2}\tilde{v}_t - \left(\gamma_v^P\right)'\boldsymbol{\varphi}_t - \left(\gamma_{Xv}^P\right)'\left(\boldsymbol{\varphi}_t^X\right) - \left(\gamma_v^I\right)'\left(\boldsymbol{\psi}_t - \beta E_t\boldsymbol{\psi}_{t+1}\right) - \left(\gamma_{vX}^I\right)'\left(\boldsymbol{\psi}_t^X - \beta E_t\boldsymbol{\psi}_{t+1}^X\right) = 0 \\ [\boldsymbol{\pi}_t] & \Gamma_\pi d(\boldsymbol{\theta})\, d(\boldsymbol{\kappa})^{-1} d(\boldsymbol{\psi})\,\boldsymbol{\pi}_t + d(\boldsymbol{\kappa})^{-1}\left(\boldsymbol{\varphi}_t - \boldsymbol{\varphi}_{t-1}\right) + \boldsymbol{\psi}_t = 0 \\ \left[\boldsymbol{\pi}_t^X\right] & \Gamma_\pi d(\boldsymbol{\theta})\, d(\boldsymbol{\kappa})^{-1} d(\boldsymbol{\psi})\, d(\boldsymbol{\phi}_x)\, d(\boldsymbol{\phi}_c)\,\boldsymbol{\pi}_t^X + d(\boldsymbol{\kappa})^{-1}\left(\boldsymbol{\varphi}_t^X - \boldsymbol{\varphi}_{t-1}^X\right) + \boldsymbol{\psi}_t^X = 0 \\ [\pi_t] & \mathbf{1}_{1\times S}\boldsymbol{\psi}_t + \mathbf{1}_{1\times S}\boldsymbol{\psi}_t^X = 0 \end{cases} \tag{35}$$

where $\boldsymbol{\varphi}_t, \boldsymbol{\varphi}_t^X, \boldsymbol{\psi}_t, \boldsymbol{\psi}_t^X$ are $S$ dimensional Lagrange multipliers for the Phillips curves and the identity.

I first focus on the long-run expectation. Assuming the existence of long-run expectations of $\tilde{v}_t = v_t - N\xi_t$ denoted by $\tilde{v}_t^\infty := \lim_{T\to\infty} E_t\tilde{v}_T$, Lagrange multipliers also have long-run expectations $\boldsymbol{\varphi}_t^\infty, \boldsymbol{\varphi}_t^{X\infty}, \boldsymbol{\psi}_t^\infty, \boldsymbol{\psi}_t^{X\infty}$.

$$\begin{cases} [\tilde{v}_t] & \Gamma_{v2}\tilde{v}_t^\infty - \left(\gamma_v^P\right)'\boldsymbol{\varphi}_t^\infty - \left(\gamma_{Xv}^P\right)'\left(\boldsymbol{\varphi}_t^{X\infty}\right) - (1-\beta)\left(\gamma_v^I\right)'\boldsymbol{\psi}_t^\infty - (1-\beta)\left(\gamma_{vX}^I\right)'\boldsymbol{\psi}_t^{X\infty} = 0 \\ [\boldsymbol{\pi}_t] & \Gamma_\pi d(\boldsymbol{\theta})\, d(\boldsymbol{\kappa})^{-1} d(\boldsymbol{\psi})\,\boldsymbol{\pi}_t^\infty + \boldsymbol{\psi}_t^\infty = 0 \\ \left[\boldsymbol{\pi}_t^X\right] & \Gamma_\pi d(\boldsymbol{\theta})\, d(\boldsymbol{\kappa})^{-1} d(\boldsymbol{\psi})\, d(\boldsymbol{\phi}_x)\, d(\boldsymbol{\phi}_c)\,\boldsymbol{\pi}_t^{X\infty} + \boldsymbol{\psi}_t^{X\infty} = 0 \\ [\pi_t] & \mathbf{1}_{1\times S}\boldsymbol{\psi}_t + \mathbf{1}_{1\times S}\boldsymbol{\psi}_t^X = 0 \end{cases}$$

Combining this with the conditions implied by the constraints in Lemma 4,

$$\begin{cases} (1-\beta)\, d(\boldsymbol{\kappa})^{-1}\boldsymbol{\pi}_t^\infty = \gamma_v^P\tilde{v}_t^\infty \\ (1-\beta)\, d(\boldsymbol{\kappa})^{-1}\boldsymbol{\pi}_t^{X\infty} = \gamma_{Xv}^P\tilde{v}_t^\infty \\ \boldsymbol{\pi}_t^\infty = \mathbf{1}_{S\times 1}\pi_t^\infty \\ \boldsymbol{\pi}_t^{X\infty} = \mathbf{1}_{S\times 1}\pi_t^\infty \end{cases},$$

the long-run expectation of the Lagrange multipliers for the Phillips curves $\varphi_t^\infty, \varphi_t^{X\infty}$ can be shown to be zeros.

Specifically, from the last three equations of the first-order conditions and the third and fourth equations of the constraints, we have $\pi_{s,t}^\infty = \pi_{s,t}^{X\infty} = \pi_t^\infty = \psi_{st}^\infty = \psi_{st}^{X\infty} = 0\ \forall s \in S$. Thus, the system simplifies to

$$\Gamma_{v2}\tilde{v}_t^\infty - \left(\gamma_v^P\right)'\boldsymbol{\varphi}_t^\infty - \left(\gamma_{Xv}^P\right)'\left(\boldsymbol{\varphi}_t^{X\infty}\right) + \psi_t^{N\infty}\left(\gamma_v^I\right)'\boldsymbol{\psi} = 0$$

and

$$\begin{cases} \gamma_v^P \tilde{v}_t^\infty = 0 & [\boldsymbol{\varphi}_t] \\ \gamma_{Xv}^P \tilde{v}_t^\infty = 0 & \left[\boldsymbol{\varphi}_t^X\right] \end{cases}.$$

That is,

$$\tilde{v}_t^\infty = 0$$

and

$$\left(\gamma_v^P\right)' \boldsymbol{\varphi}_t^\infty + \left(\gamma_{Xv}^P\right)' \boldsymbol{\varphi}_t^{X\infty} - \psi_t^{N\infty} \left(\gamma_v^I\right)' \boldsymbol{\psi} = \Gamma_{v2} \tilde{v}_t^\infty$$

$$\begin{bmatrix} \left(\gamma_v^P\right)' & \left(\gamma_{Xv}^P\right)' & \left(\gamma_v^I\right)' \boldsymbol{\psi} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varphi}_t^\infty \\ \boldsymbol{\varphi}_t^{X\infty} \\ -\psi_t^{N\infty} \end{bmatrix} = 0$$

Next, by summing the first-order conditions (35) with respect to sectoral inflation rates $\boldsymbol{\pi}_t, \boldsymbol{\pi}_t^X$, I obtain

$$\Gamma_\pi \sum_{s\in S} \frac{\theta_s}{\kappa_s} \psi_s \left[\pi_{s,t} + \frac{\phi_{sx}}{\phi_{sc}} \pi_{s,t}^X\right] + \sum_{s\in S} \left(\kappa_s^{-1} \left(\varphi_{st} - \varphi_{st-1}\right) + \kappa_s^{-1} \left(\varphi_{st}^X - \varphi_{st-1}^X\right)\right) = 0.$$

Recalling the definitions of $\pi_{s,t}$ and $\pi_{s,t}^X$,

$$\Gamma_\pi \sum_{s\in S} \frac{\theta_s}{\kappa_s} \psi_s \left[\left(\log P_{st} - \log P_{st-1}\right) + \frac{\phi_{sx}}{\phi_{sc}} \left(\log P_{st}^X - \log P_{st-1}^X\right)\right] + \sum_{s\in S} \left(\kappa_s^{-1} \left(\varphi_{st} - \varphi_{st-1}\right) + \kappa_s^{-1} \left(\varphi_{st}^X - \varphi_{st-1}^X\right)\right) = 0.$$

By rearranging, we can see that for any $t$,

$$\sum_{s\in S} \left[\Gamma_\pi \frac{\theta_s}{\kappa_s} \psi_s \left(\log P_{st} + \frac{\phi_{sx}}{\phi_{sc}} \log P_{st}^X\right) + \kappa_s^{-1} \varphi_{st} + \kappa_s^{-1} \varphi_{st}^X\right] = const.$$

This also holds in long-run expectation.

Since the long-run expectations of the Phillips curve Lagrange multipliers are zero, we obtain

$$\lim_{T\to\infty} E_t \sum_{s\in S} \frac{\theta_s}{\kappa_s} \psi_s \left[\log P_{sT} + \frac{\phi_{sx}}{\phi_{sc}} \log P_{sT}^X\right] = \overline{\log \mathbb{P}},$$

where $\overline{\log \mathbb{P}}$ is a constant.

# C Appendix to Section 4

## C.1 Detailed welfare evaluation procedure

For each country-specific calibration of these parameters, we can solve for the equilibrium characterized by the Phillips curves, identities relating inflation rates and relative prices, and the normalization of CPI and monetary policy.

$$\begin{cases} \frac{\lambda_s}{(1-\lambda_s)(1-\lambda_s\beta)}\left(\pi_{s,t}-\beta E_t\left[\pi_{s,t+1}\right]\right)=\alpha_{sm}\left(q_t+q^*_{st}\right)+\alpha_{sl}w_t-z_{st}-p_{st} & \forall s \\ \frac{\lambda_s}{(1-\lambda_s)(1-\lambda_s\beta)}\left(\pi^X_{s,t}-\beta E_t\left[\pi^X_{s,t+1}\right]\right)=\alpha_{sm}\left(q_t+q^*_{st}\right)+\alpha_{sl}w_t-z_{st}-p^X_{st} & \forall s \\ \pi_{s,t}=\pi_t+p_{st}-p_{st-1} & \forall s \\ \pi^X_{s,t}=\pi_t+p^X_{st}-p^X_{st-1} & \forall s \\ Monetary\ Policy \end{cases}.$$

The normalization equation comes from all nominal variables being expressed relative to CPI.

To consider the optimal policy, denote $y_t=\left[c_{1t},...,c_{St},x_{1t},...,x_{St},\pi_{1t},...,\pi_{St},\pi^X_{1t},...,\pi^X_{St},\pi_t\right]$ , $\xi_t=\left[c^*_t,x^*_{1t},...,x^*_{St},p^*_{1t},...,p^*_{St},q^*_{1t},...,q^*_{St},z_{1t}...,,z_{St}\right]'$. Define $\Gamma_{y2},\Gamma_{\xi y},\gamma^P_y,\gamma^P_{yp},\gamma^P_x,\gamma^I_y,\gamma^I_{ym}$ so that

$$L^{-(1+\phi)}\left(\mathcal{W}-\bar{\mathcal{W}}\right)|_{\text{efficient}}=E_0\sum_{t=0}^{\infty}\left[y_t'\Gamma_{y2}y_t+\xi_t'\Gamma_{\xi y}y_t\right]$$

$$\begin{bmatrix} \left(-\frac{\lambda_s}{(1-\lambda_s)(1-\lambda_s\beta)}\left(\pi_{s,t}-\beta E_t\left[\pi_{s,t+1}\right]\right)+\alpha_{sm}\left(q_t+q^*_{st}\right)+\alpha_{sl}w_t-z_{st}-p_{st}\right)_{s\in S} \\ \left(-\frac{\lambda_s}{(1-\lambda_s)(1-\lambda_s\beta)}\left(\pi^X_{s,t}-\beta E_t\left[\pi^X_{s,t+1}\right]\right)+\alpha_{sm}\left(q_t+q^*_{st}\right)+\alpha_{sl}w_t-z_{st}-p^X_{st}\right)_{s\in S} \end{bmatrix}=\gamma^P_y y_t+\gamma^P_{yp}E_t y_{t+1}+\gamma^P_x\xi_t$$

$$\begin{bmatrix} -\pi_{s,t}+\pi_t+p_{st}-p_{st-1} \\ -\pi^X_{s,t}+\pi_t+p^X_{st}-p^X_{st-1} \end{bmatrix}=\gamma^I_y y_t+\gamma^I_{ym}y_{t-1}.$$

Consider

$$\max E_0\sum_{t=0}^{\infty}\left[y_t'\Gamma_{y2}y_t+\xi_t'\Gamma_{\xi y}y_t\right]$$

$$s.t.\begin{cases}\gamma^P_y y_t+\gamma^P_{yp}E_t y_{t+1}+\gamma^P_x\xi_t=0 & \varphi^P_t \\ \gamma^I_y y_t+\gamma^I_{ym}y_{t-1}=0 & \varphi^I_t\end{cases}$$

The first-order condition is

$$2\Gamma_{y2}y_t+\Gamma'_{\xi y}\xi_t+\gamma^{P\prime}_y\varphi^P_t+\gamma^{I\prime}_y\varphi^I_t+\gamma^{P\prime}_{yp}\varphi^P_{t-1}+\gamma^{I\prime}_{ym}E_t\varphi^I_{t+1}=0.$$

Thus, assuming an exogenous process $\xi_{t+1}=\rho\xi_t+u_t$, I solve the dynamics

$$\begin{cases}\gamma^P_y y_t+\gamma^P_{yp}E_t y_{t+1}+\gamma^P_x\xi_t=0 \\ \gamma^I_y y_t+\gamma^I_{ym}y_{t-1}=0 \\ E_t\xi_{t+1}-\rho\xi_t=0 \\ 2\Gamma_{y2}y_t+\Gamma'_{\xi y}\xi_t+\gamma^{P\prime}_y\varphi^P_t+\gamma^{I\prime}_y\varphi^I_t+\gamma^{P\prime}_{yp}\varphi^P_{t-1}+\gamma^{I\prime}_{ym}E_t\varphi^I_{t+1}=0\end{cases}$$

and evaluate welfare at the solution.

It is convenient to define $\tilde{y}_t=\left[y_t',\left(\varphi^P_t\right)',\left(\varphi^I_t\right)'\right]'$ and $x_t=\left[\xi_t',\left(p_{t-1}\right)',\left(p^X_{t-1}\right)',\left(\varphi^P_{t-1}\right)'\right]$, $\tilde{u}_t=\left[u_t',\mathbb{O}_{1\times 3S}\right]'$. Then, the solution takes the form

$$\tilde{y}_t=G_x x_t$$

$$x_{t+1}=H_x x_t+\tilde{u}_t.$$

Note that $H_x$ consists of two parts, one without the Lagrange multipliers and the Lagrange multipliers.

$$\begin{bmatrix} \begin{bmatrix} \xi_{t+1} \\ \hat{p}_t \\ \hat{p}_t^X \end{bmatrix} \\ \varphi_t^P \end{bmatrix} = \begin{bmatrix} H_{xx} & \mathbb{O} \\ H_{\varphi x} & H_{\varphi\varphi} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \xi_t \\ p_{t-1} \\ p_{t-1}^X \end{bmatrix} \\ \varphi_{t-1}^P \end{bmatrix} + \begin{bmatrix} I & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} u_{t+1}.$$

### C.1.1 Alternative policies

Alternative policies can be solved for by replacing the first-order condition with the monetary policy rule considered.

I also track the Lagrange multipliers $\varphi_t^P, \varphi_t^I$ defined as in the optimal dynamics as auxiliary variables that do not affect the system (that is, defined by the state variable $x_t$ and do not appear in any of the other equations). To do so, I solve

$$\begin{cases} \gamma_y^P y_t + \gamma_{yp}^P E_t y_{t+1} + \gamma_x^P \xi_t = 0 \\ \gamma_y^I y_t + \gamma_{ym}^I y_{t-1} = 0 \\ E_t \xi_{t+1} - \rho \xi_t = 0 \\ \pi_t = 0 \text{ or } \quad \sum_{s \in S} \mathbb{I}_{s \in Core} \psi_s \pi_{st} = 0 \\ -\varphi_t^P + H_\varphi x_t = 0 \end{cases}.$$

In this way, the solution takes the same form

$$\tilde{y}_t = G_x x_t$$

$$x_{t+1} = H_x x_t + \tilde{u}_t.$$

Note that the difference in the policy is reflected in the coefficients $G_x$ and $H_x$.

### C.1.2 Calculation of welfare

The unconditional expectation of welfare

$$E\left[L^{-(1+\phi)}\left(\mathcal{W} - \bar{\mathcal{W}}\right)|_{\text{efficient}}\right] = E\sum_{t=0}^{\infty}\left[y_t'\Gamma_{y2}y_t + \xi_t'\Gamma_{\xi y}y_t\right] = E\sum_{t=0}^{\infty}\left[\tilde{y}_t'\tilde{\Gamma}_{y2}\tilde{y}_t + x_t'\tilde{\Gamma}_{\xi y}\tilde{y}_t\right]$$

under any solution

$$\tilde{y}_t = G_x x_t$$

$$x_{t+1} = H_x x_t + \tilde{u}_t$$

can be calculated as follows by assuming $E\tilde{u}_t\tilde{u}_t' = \Sigma_u, E\tilde{u}_t\tilde{u}_s' = 0\ \forall t \neq s$. Define

$$V = \frac{\beta}{1-\beta}\Sigma_u + \underbrace{Ex_0x_0'}_{=:\Sigma_x} + \beta H_x V H_x',$$

then

$$E\left[L^{-(1+\phi)}\left(\mathcal{W} - \bar{\mathcal{W}}\right)|_{\text{efficient}}\right] = tr\left[\left(G_x'\Gamma_{2y}G_x + 2\Gamma_{yx}G_x\right)V\right].$$

The choice of $\Sigma_x$ depends on the type of policy experiment.

I consider two types of policy experiment. The first type is that in which the economy starts form the stationary distribution obtained under headline inflation targeting as an approximation of the current policy. Then, this experiment compares switching from the current headline inflation targeting to different policies. To obtain the variance-covariance matrix, I use $H_x$ obtained under the headline targeting policy $H_x^{Head}$. By solving

$$x_{t+1} = H_x^{Head} x_t + u_{t+1},$$

we obtain

$$\Sigma_x = H_x^{Head} \Sigma_x \left( H_x^{Head} \right)' + \Sigma_u.$$

The second type of policy experiment compares different worlds each of which starts from the steady state under the policy considered and continues the policy. In this case, $\Sigma_x$ is the solution to

$$\Sigma_x = H_x \Sigma_x \left( H_x \right)' + \Sigma_u$$

where $H_x$ is the solution to the equilibrium system under each policy.

### C.1.3 Conversion to units of consumption

To interpret the welfare loss in units of consumption, the following procedure calculates the consumption equivalent of the welfare loss relative to the optimal policy. Compare the welfare at the optimal

$$W^O := E_0 \sum_{t=0}^{\infty} \beta^t \left[ \frac{\left(C_t^O\right)^{1-\sigma}}{1-\sigma} - \frac{\left(L_t^O\right)^{1+\phi}}{1+\phi} \right] + \Lambda D_0^O$$

with sub-optimal

$$W^S := E_0 \sum_{t=0}^{\infty} \beta^t \left[ \frac{\left(C_t^S\right)^{1-\sigma}}{1-\sigma} - \frac{\left(L_t^S\right)^{1+\phi}}{1+\phi} \right] + \Lambda D_0^S.$$

Consider discounting $C_t^O$ by a fraction $\gamma^S$ to make them equal.

$$W^S = E_0 \sum_{t=0}^{\infty} \beta^t \left[ \frac{\left(\left(1-\gamma^S\right) C_t^O\right)^{1-\sigma}}{1-\sigma} - \frac{\left(L_t^O\right)^{1+\phi}}{1+\phi} \right] + \Lambda D_0^O$$

Using the approximation, $U_t \approx U + C^{1-\sigma}\left(\hat{c}_t + \frac{1-\sigma}{2}\hat{c}_t^2\right) - L^{1+\phi}\left(\hat{l}_t + \frac{1+\phi}{2}\hat{l}_t^2\right)$

$$W^S = E_0 \sum_{t=0}^{\infty} \beta^t C^{1-\sigma} \left( \log\left(1-\gamma^S\right) + \frac{1-\sigma}{2}\left(\log\left(1-\gamma^S\right)\right)^2 + (1-\sigma)\log\left(1-\gamma^S\right)\hat{c}_t^O \right) + W^O$$

Under the stationarity of exogenous variables, $E_0 \hat{c}_t = 0$. Thus,

$$\log\left(1-\gamma^S\right) + \frac{1-\sigma}{2}\left(\log\left(1-\gamma^S\right)\right)^2 = \frac{(1-\beta)}{\sum_{s\in S} \phi_{sc} \frac{\phi_{ls}}{\alpha_{sl}}} \left( \frac{W^S}{L^{1+\phi}} - \frac{W^O}{L^{1+\phi}} \right)$$

Table 6: Concordance between WIOT, NS2008, and BW2006

| WIOT | description | ISIC | NS2008 |
| --- | --- | --- | --- |
| c1 | Agriculture, Hunting, Forestry and Fishing | 01, 02, 05 | Farm products |
| c2 | Mining and Quarrying | 10-14 | (Note 1) |
| c3 | Food, Beverages and Tobacco | 15,16 | Processed foods and feeds |
| c4 | Textiles and Textile Products | 17,18 | Textile products and apparel |
| c5 | Leather, Leather and Footwear | 19 | Hides, skins, leather, and related products |
| c6 | Wood and Products of Wood and Cork | 20 | Lumber and wood products |
| c7 | Pulp, Paper, Paper , Printing and Publishing | 21,22 | Pulp, paper, and allied products |
| c8 | Coke, Refined Petroleum and Nuclear Fuel | 23 | Fuels and related products and power |
| c9 | Chemicals and Chemical Products | 24 | Chemicals and allied products |
| c10 | Rubber and Plastics | 25 | Rubber and plastic products |
| c11 | Other Non-Metallic Mineral | 26 | Nonmetallic mineral products |
| c12 | Basic Metals and Fabricated Metal | 27,28 | Metals and metal products |
| c13 | Machinery, Nec | 29 | Machinery and equipment |
| c14 | Electrical and Optical Equipment | 30-33 | Machinery and equipment |
| c15 | Transport Equipment | 34,35 | Transportation equipment |
| c16 | Manufacturing, Nec; Recycling | 36,37 | Miscellaneous products |
| c17 | Electricity, Gas and Water Supply | 40,41 | Fuels and related products and power |
| c18 | Construction | 45 | Services (excl. travel) |
| c19 | Sale, Maintenance and Repair of Motor Vehicles and Motorcycles; Retail Sale of Fuel | 50 | (Note 2) |
| c20 | Wholesale Trade and Commission Trade, Except of Motor Vehicles and Motorcycles | 51 | Services (excl. travel) |
| c21 | Retail Trade, Except of Motor Vehicles and Motorcycles; Repair of Household Goods | 52 | Services (excl. travel) |
| c22 | Hotels and Restaurants | 55 | Services (excl. travel) |
| c23 | Inland Transport | 60 | Travel |
| c24 | Water Transport | 61 | Travel |
| c25 | Air Transport | 62 | Travel |
| c26 | Other Supporting and Auxiliary Transport Activities; Activities of Travel Agencies | 63 | Travel |
| c27 | Post and Telecommunications | 64 | Services (excl. travel) |
| c28 | Financial Intermediation | 65-67 | Services (excl. travel) |
| c29 | Real Estate Activities | 70 | Services (excl. travel) |
| c30 | Renting of M and Eq and Other Business Activities | 71-74 | Services (excl. travel) |
| c31 | Public Admin and Defence; Compulsory Social Security | 75 | Services (excl. travel) |
| c32 | Education | 80 | Services (excl. travel) |
| c33 | Health and Social Work | 85 | Services (excl. travel) |
| c34 | Other Community, Social and Personal Services | 90-93 | Services (excl. travel) |
| c35 | Private Households with Employed Persons | 95 | Services (excl. travel) |

$$\Rightarrow \gamma^S = 1 - \exp\left\{ \frac{-1 + \sqrt{1 + 2\left(1-\sigma\right)\frac{(1-\beta)}{\sum_{s\in S}\phi_{sc}\frac{\phi_{ls}}{\alpha_{sl}}}\left(\frac{W^S}{L^{1+\phi}} - \frac{W^O}{L^{1+\phi}}\right)}}{1-\sigma} \right\}.$$

## C.2 Concordance of sectors across the World Input-Output Table, Nakamura and Steinsson (2008) and Broda and Weinstein (2006)

Table 6 is the concordance table created by the author.

## C.3 Input-output adjustment

By aggregating the input-output table, I can obtain the following matrix.

$$\begin{bmatrix} P_1 Y_{11} & \cdots & P_1 Y_{1S} & P_1 C_1 & P_1^X X_1 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ P_S Y_{S1} & \cdots & P_S Y_{SS} & P_S C_S & P_S^X X_S \\ \mathcal{E} Q_1^* M_1 & \cdots & \mathcal{E} Q_S^* M_S & n.a. & n.a. \\ W L_1 & \cdots & W L_S & n.a. & n.a. \end{bmatrix}$$

Define

$$Totc_s = \sum_{s'} P_{s'} Y_{s's} + \mathcal{E} Q_s^* M_s + W L_s$$

$$\tilde{\alpha}_{ls} = \frac{W L_s}{Totc_s},\ \tilde{\alpha}_{ms} = \frac{\mathcal{E} Q_s^* M_s}{Totc_s},\ A = \begin{bmatrix} \frac{P_1 Y_{11}}{Totc_1} & \cdots & \frac{P_1 Y_{1S}}{Totc_S} \\ \vdots & \ddots & \vdots \\ \frac{P_S Y_{S1}}{Totc_1} & \cdots & \frac{P_S Y_{SS}}{Totc_S} \end{bmatrix}$$

Then, if we count all indirect usage of labor and imported goods,

$$\begin{bmatrix} \alpha_{l1} \\ \vdots \\ \alpha_{lS} \end{bmatrix} = \begin{bmatrix} \tilde{\alpha}_{l1} \\ \vdots \\ \tilde{\alpha}_{lS} \end{bmatrix} + A' \begin{bmatrix} \tilde{\alpha}_{l1} \\ \vdots \\ \tilde{\alpha}_{lS} \end{bmatrix} + (A')^2 \begin{bmatrix} \tilde{\alpha}_{l1} \\ \vdots \\ \tilde{\alpha}_{lS} \end{bmatrix} + .... = (I - A')^{-1} \begin{bmatrix} \tilde{\alpha}_{l1} \\ \vdots \\ \tilde{\alpha}_{lS} \end{bmatrix}.$$

$$\begin{bmatrix} \alpha_{m1} \\ \vdots \\ \alpha_{mS} \end{bmatrix} = \begin{bmatrix} \tilde{\alpha}_{m1} \\ \vdots \\ \tilde{\alpha}_{mS} \end{bmatrix} + A' \begin{bmatrix} \tilde{\alpha}_{m1} \\ \vdots \\ \tilde{\alpha}_{mS} \end{bmatrix} + (A')^2 \begin{bmatrix} \tilde{\alpha}_{m1} \\ \vdots \\ \tilde{\alpha}_{mS} \end{bmatrix} + .... = (I - A')^{-1} \begin{bmatrix} \tilde{\alpha}_{m1} \\ \vdots \\ \tilde{\alpha}_{mS} \end{bmatrix}.$$

Similarly, define

$$P_s Y_s = \sum_{s'} P_s Y_{ss'} + P_s C_s + \frac{\theta_s^* - 1}{\theta_s^*} P_s^X X_s$$

$$\tilde{\phi}_{sc} = \frac{P_s C_s}{P_s Y_s},\ \tilde{\phi}_{sx} \left( = \frac{P_s}{P_s^X} \frac{P_s^X X_s}{P_s Y_s} \right) = \frac{\theta^* - 1}{\theta^*} \frac{P_s^X X_s}{P_s Y_s},\ \Phi = \begin{bmatrix} \frac{P_1 Y_{11}}{P_1 Y_1} & \cdots & \frac{P_1 Y_{1S}}{P_1 Y_1} \\ \vdots & \ddots & \vdots \\ \frac{P_S Y_{S1}}{P_S Y_S} & \cdots & \frac{P_S Y_{SS}}{P_S Y_S} \end{bmatrix}$$

Then, I count all indirect demand from domestic and foreign consumers,

$$\begin{bmatrix} \phi_{1c} \\ \vdots \\ \phi_{Sc} \end{bmatrix} = \begin{bmatrix} \tilde{\phi}_{1c} \\ \vdots \\ \tilde{\phi}_{Sc} \end{bmatrix} + \Phi \begin{bmatrix} \tilde{\phi}_{1c} \\ \vdots \\ \tilde{\phi}_{Sc} \end{bmatrix} + \Phi^2 \begin{bmatrix} \tilde{\phi}_{1c} \\ \vdots \\ \tilde{\phi}_{Sc} \end{bmatrix} ... = (I - \Phi)^{-1} \begin{bmatrix} \tilde{\phi}_{1c} \\ \vdots \\ \tilde{\phi}_{Sc} \end{bmatrix}$$

$$\begin{bmatrix} \phi_{1x} \\ \vdots \\ \phi_{Sx} \end{bmatrix} = \begin{bmatrix} \tilde{\phi}_{1x} \\ \vdots \\ \tilde{\phi}_{Sx} \end{bmatrix} + \Phi \begin{bmatrix} \tilde{\phi}_{1x} \\ \vdots \\ \tilde{\phi}_{Sx} \end{bmatrix} + \Phi^2 \begin{bmatrix} \tilde{\phi}_{1x} \\ \vdots \\ \tilde{\phi}_{Sx} \end{bmatrix} ... = (I - \Phi)^{-1} \begin{bmatrix} \tilde{\phi}_{1x} \\ \vdots \\ \tilde{\phi}_{Sx} \end{bmatrix}$$